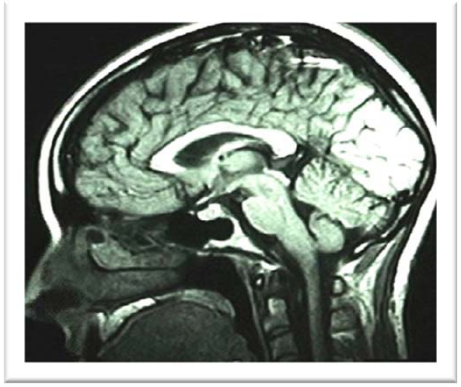
# The Sparse Fourier Transform: Applications

Haitham Hassanieh and Dina Katabi


# Fourier Transform Is Used Everywhere

**Audio & Video Compression**

**GPS** **Radar**

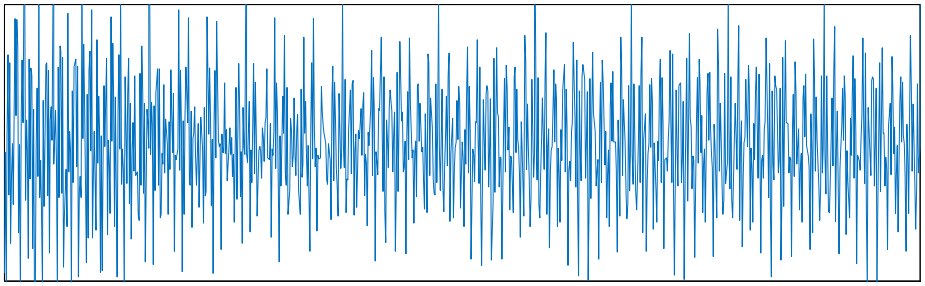


**Medical Imaging**

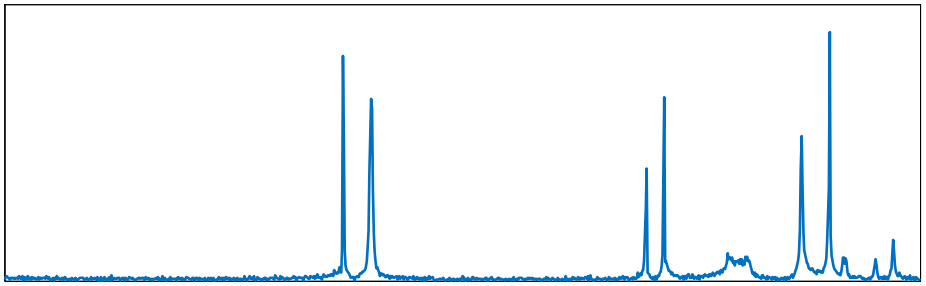
**DNA Sequencing**

**Oil Exploration**

# Can we use Sparse Fourier Transform?



Time



Frequency

**Sparse Fourier Transform: compute the Fourier transform in sublinear time faster than FFT.**

**Faster Acquisition**

**Faster Computation**

# Sparse Fourier Transform Theory

- For a signal of size $n$ with $k$ large frequencies
- Run Time Complexity
  - Exactly sparse: $O(k \log n)$
  - Approx. sparse: $O(k \log(n) \log(n/k))$

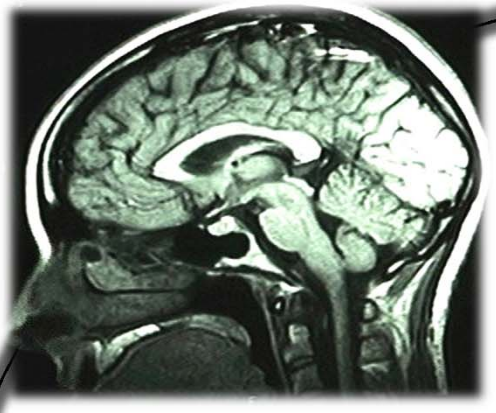
**Faster Computation**

- Sampling Complexity (Average Case)
  - Exactly sparse: $O(k)$ samples
  - Approx. sparse: $O(k \log(n))$ samples

**Faster Acquisition**

**Does it work in practice?**

# Applications of Sparse FFT

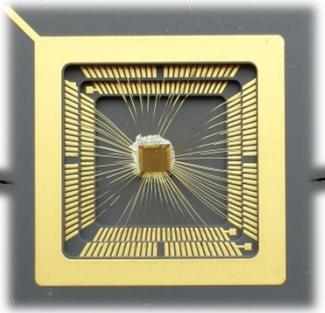


Medical Imaging

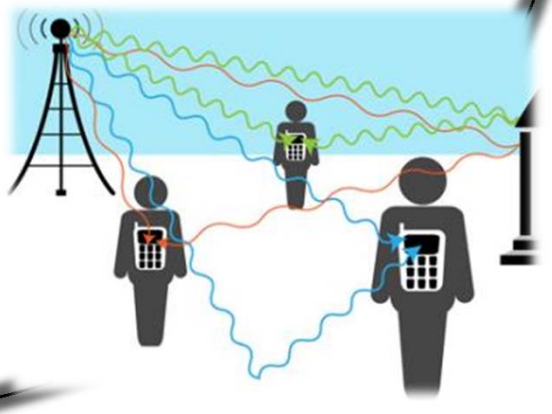
GPS

Computational Photography

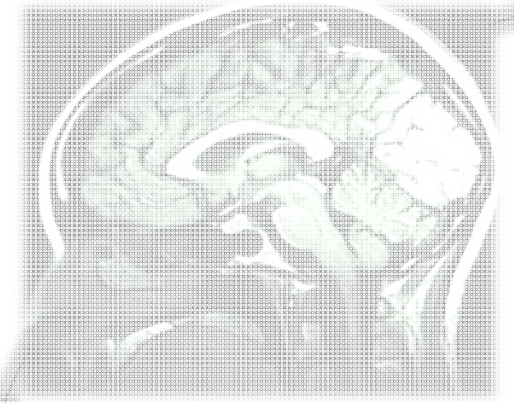
Hardware



Spectrum Sensing



# Applications of Sparse FFT



Medical Imaging

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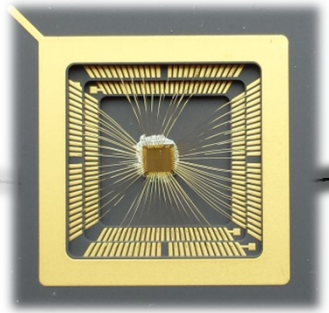
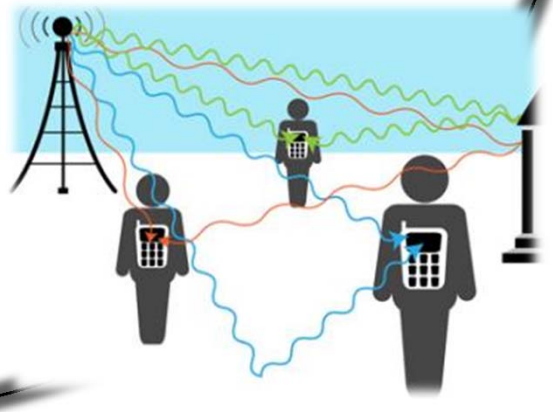
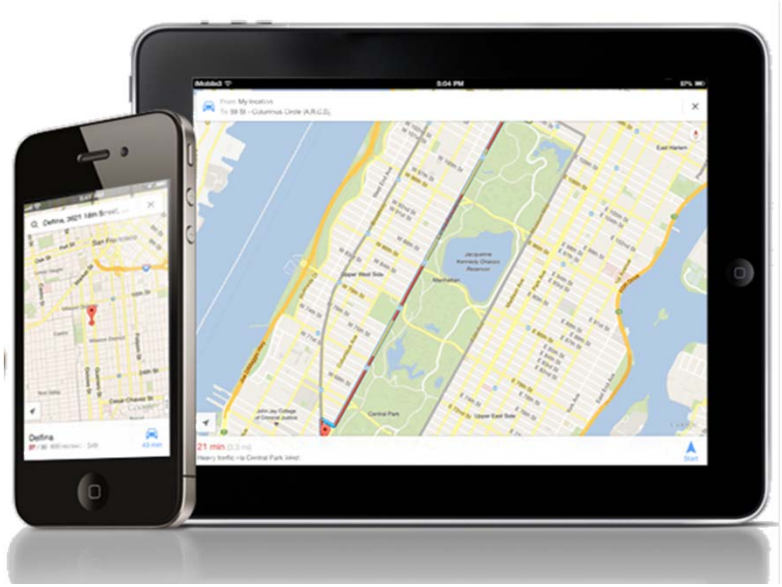


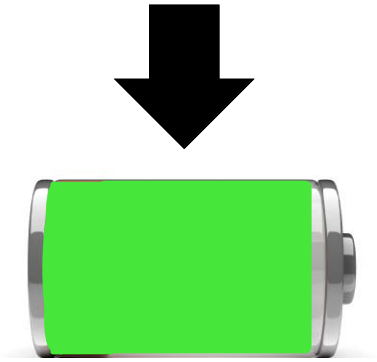
Spectrum Sensing



# GPS

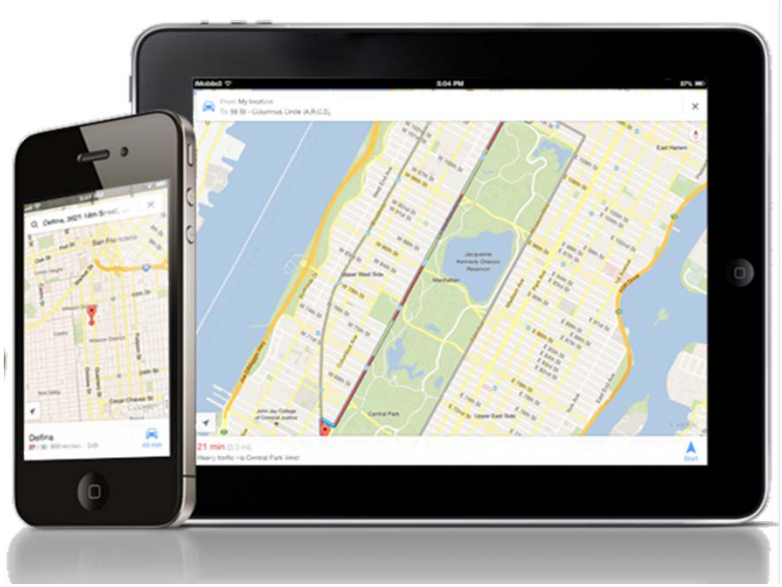


GPS consumes a lot of power



H. Hassanieh, F. Adib, D. Katabi, & P. Indyk , *Faster GPS via Sparse Fourier Transform* **MOBICOM'12**

# GPS



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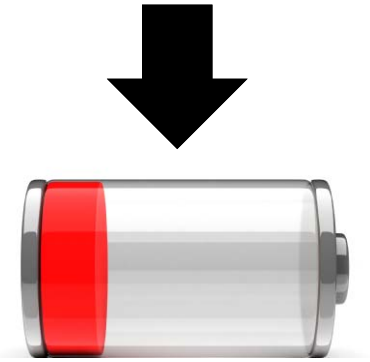


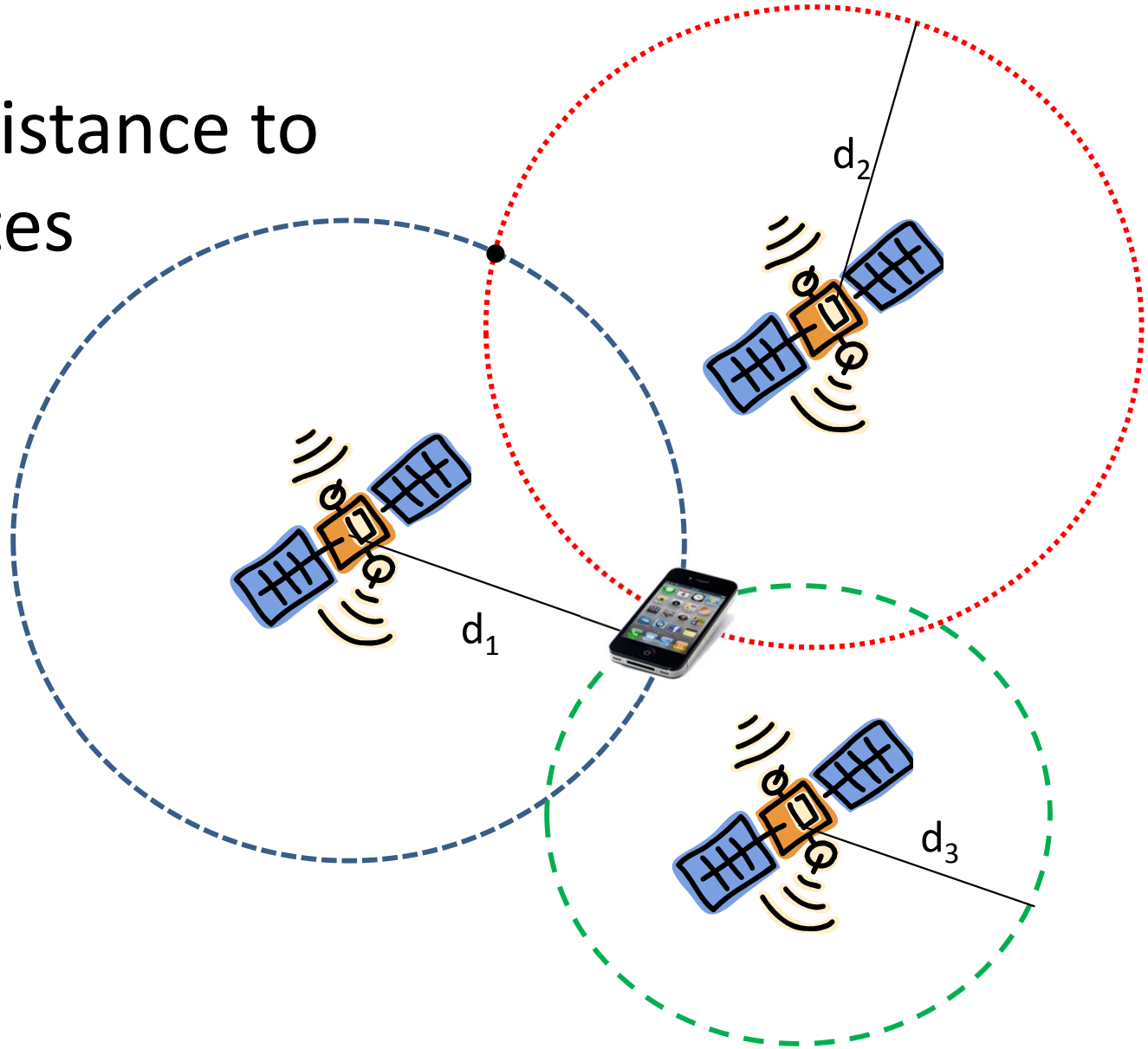
Drain Battery

## Can we use sparse FFT?

H. Hassanieh, F. Adib, D. Katabi, & P. Indyk , *Faster GPS via Sparse Fourier Transform* **MOBICOM'12**

# How Does GPS Work?

Compute the distance to the GPS satellites



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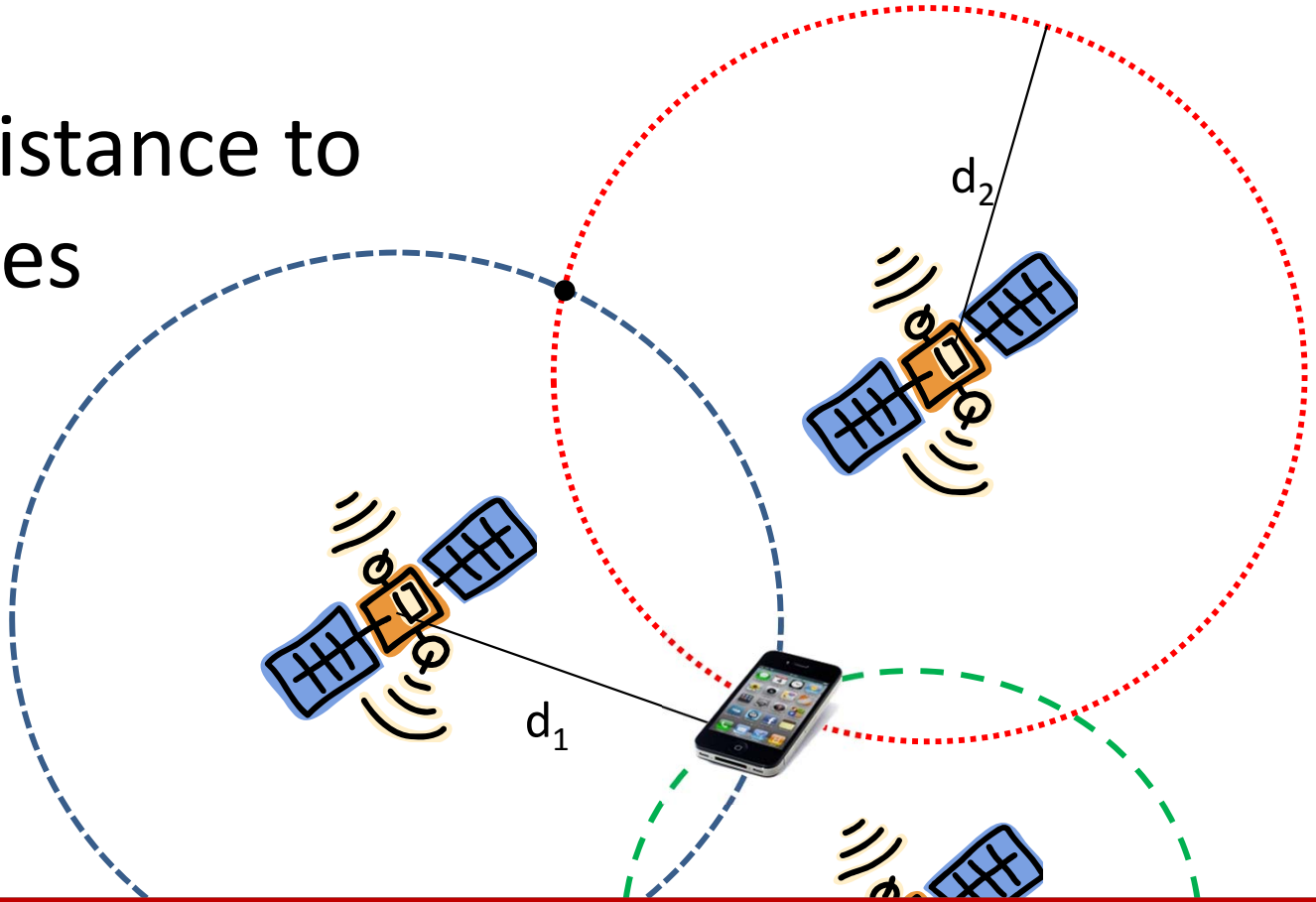
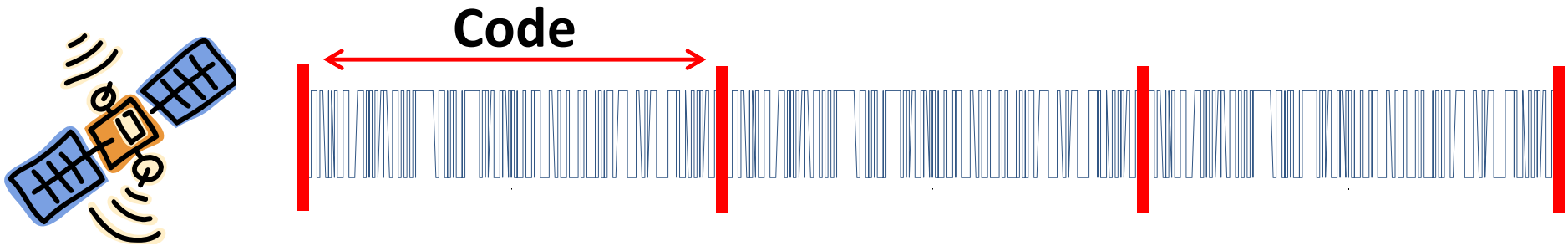



**distance = propagation delay × speed of light**

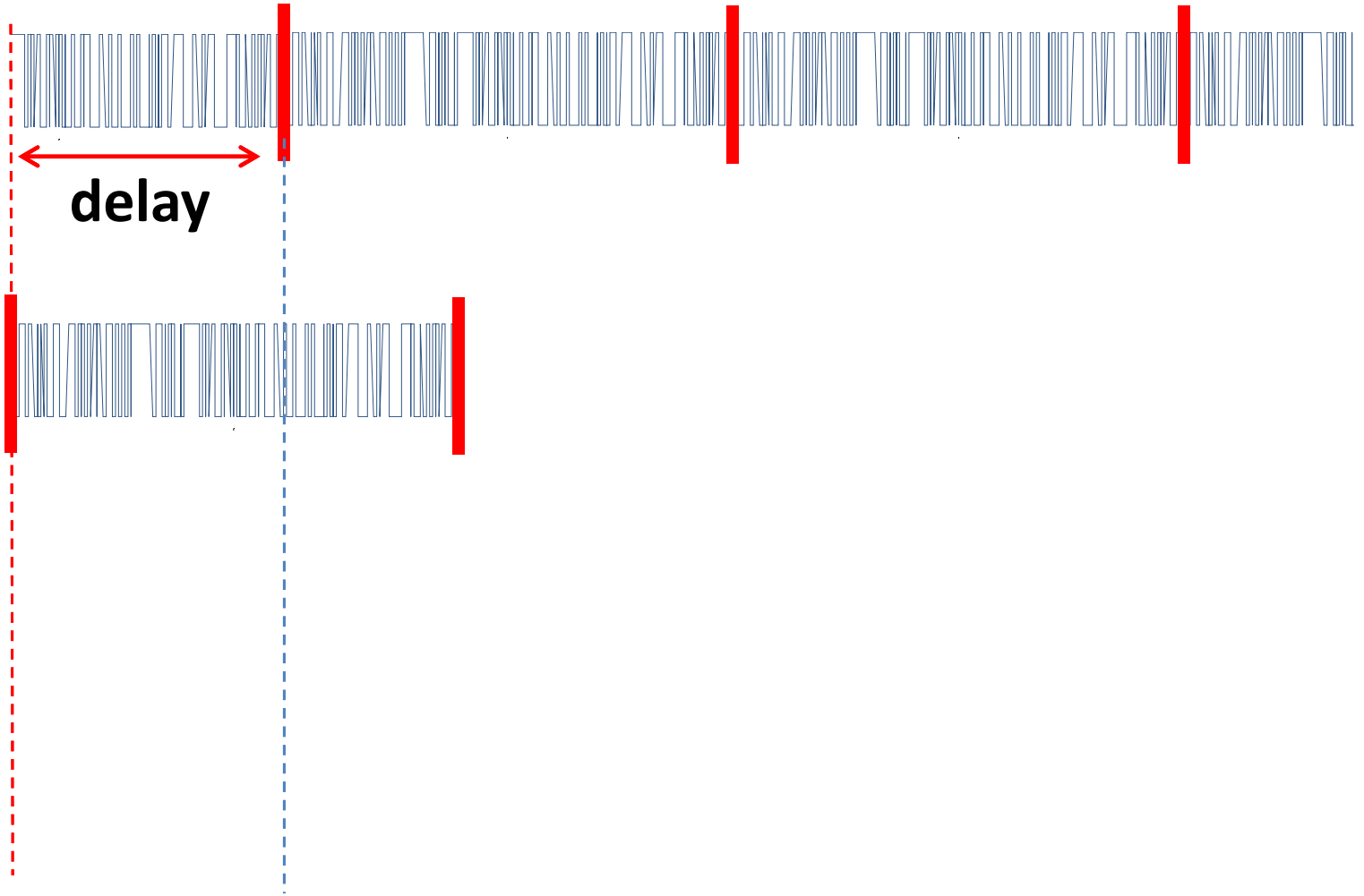
# How to Compute the Propagation Delay?



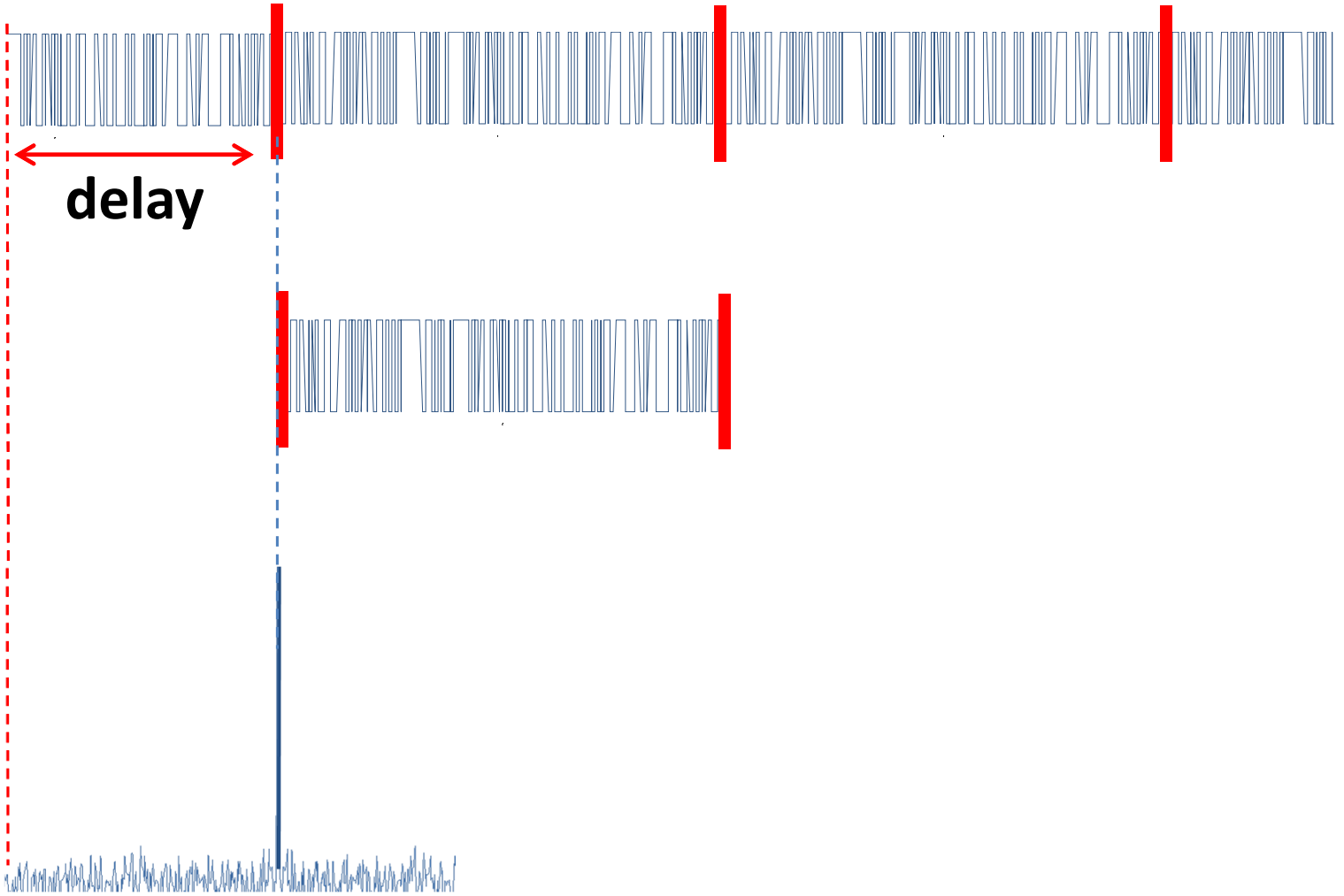
# How to Compute the Propagation Delay?


Code arrives shifted by propagation delay

# How to Compute the Propagation Delay?

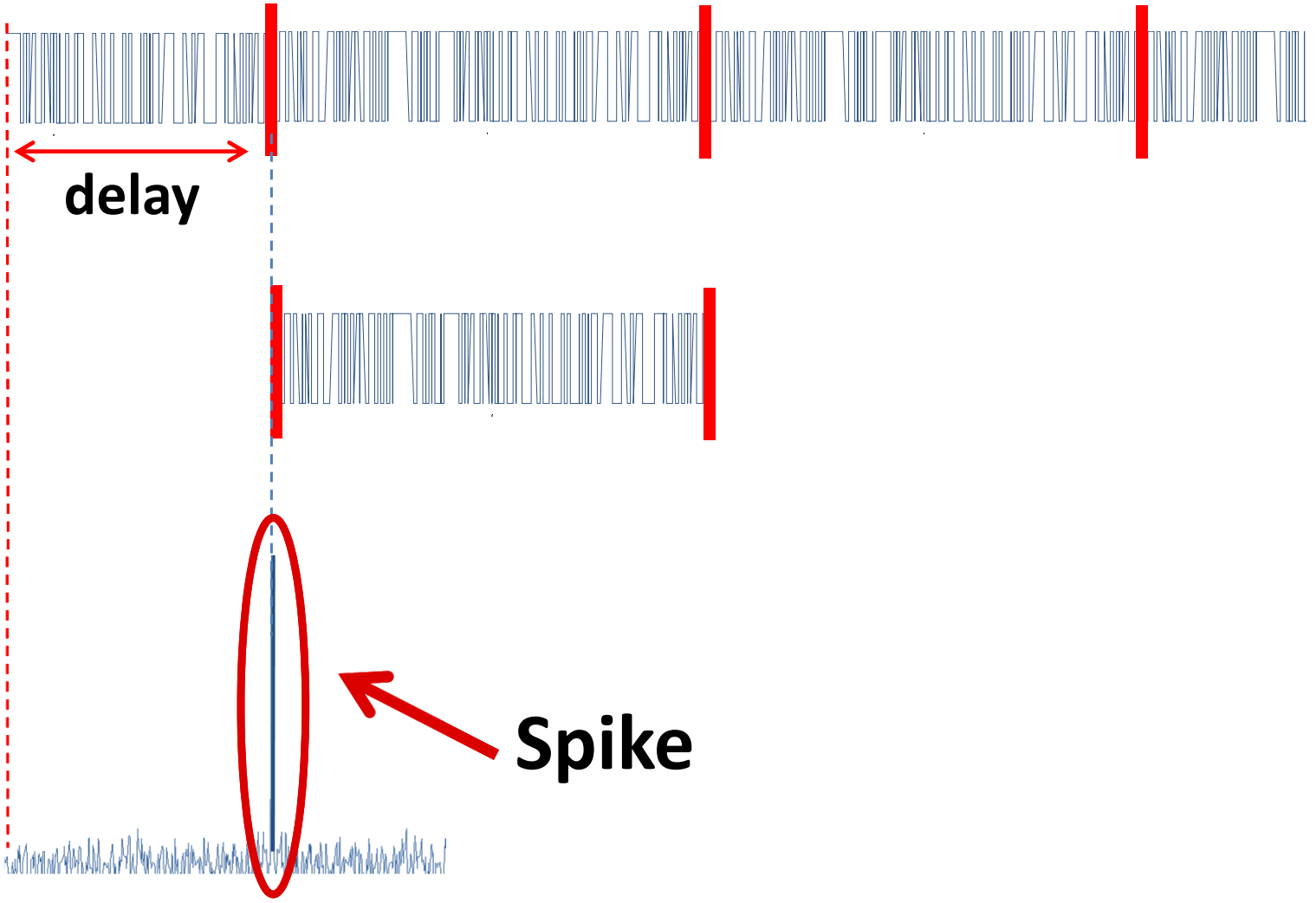


# How to Compute the Propagation Delay?




**Correlation**

# How to Compute the Propagation Delay?



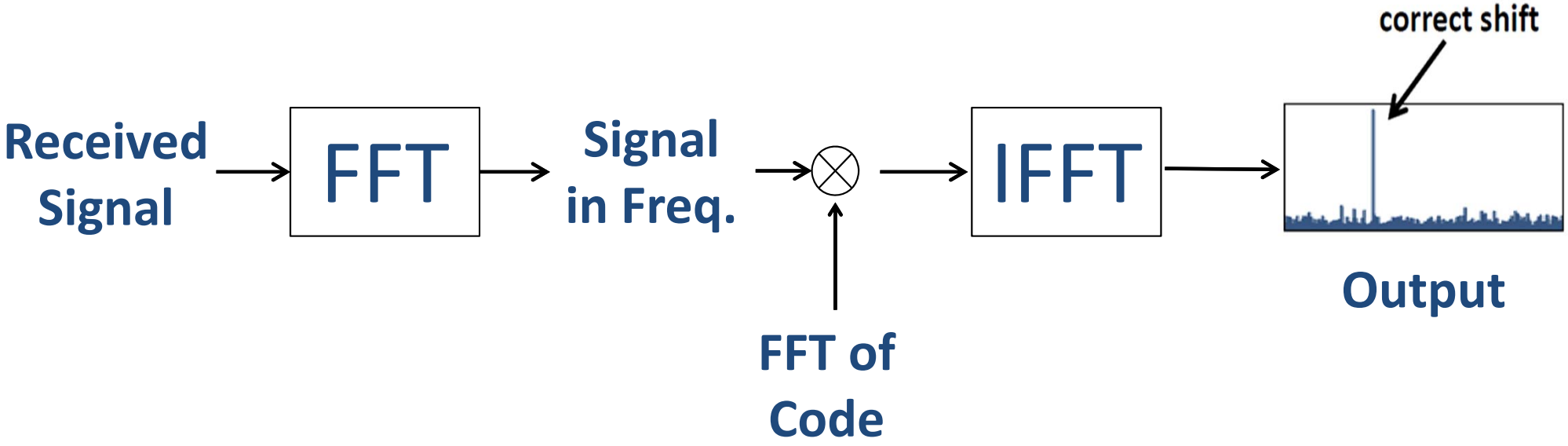

Spike determines the delay

**Convolution in Time** ⟺ **Multiplication in Frequency**

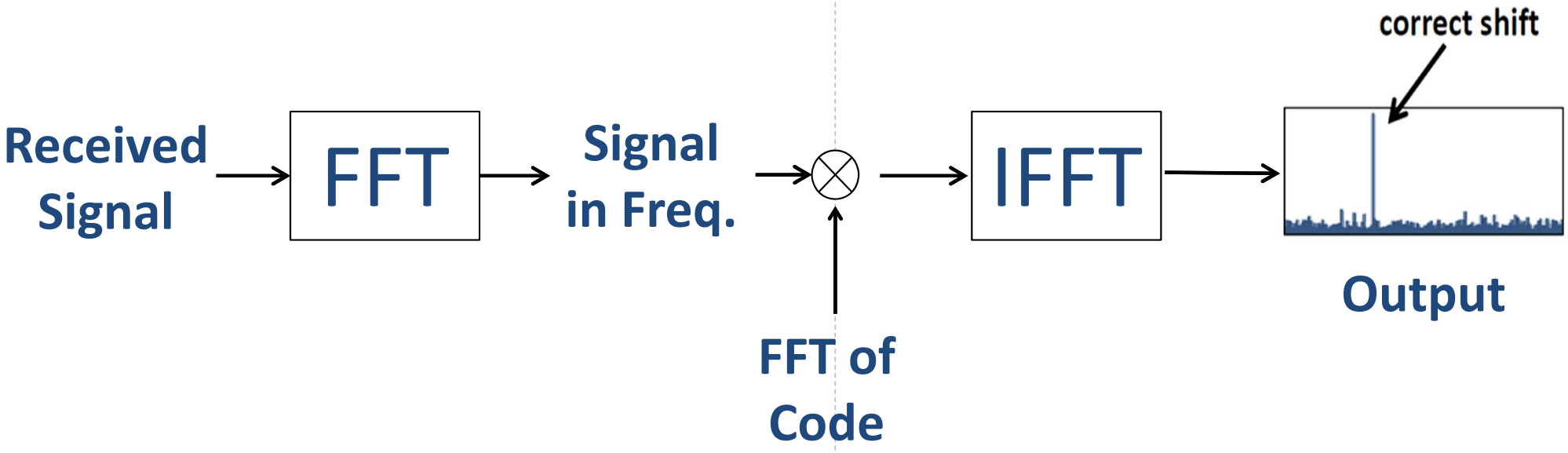
$O(n^2)$ $O(n \log n)$

$n$ : Number of samples in the code

# GPS Using FFT



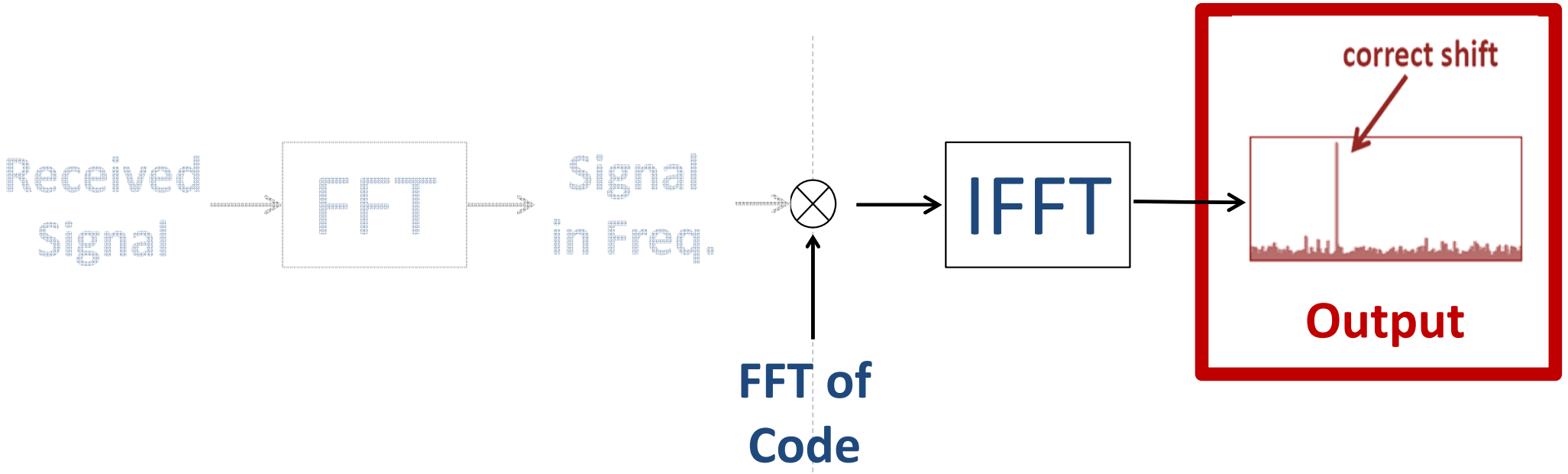
# GPS Using FFT




**Each stage takes $O(n \log n)$**

**→ need to reduce complexity of both stages**

# GPS Using FFT




FFT Stage

IFFT Stage

**Sparse IFFT**

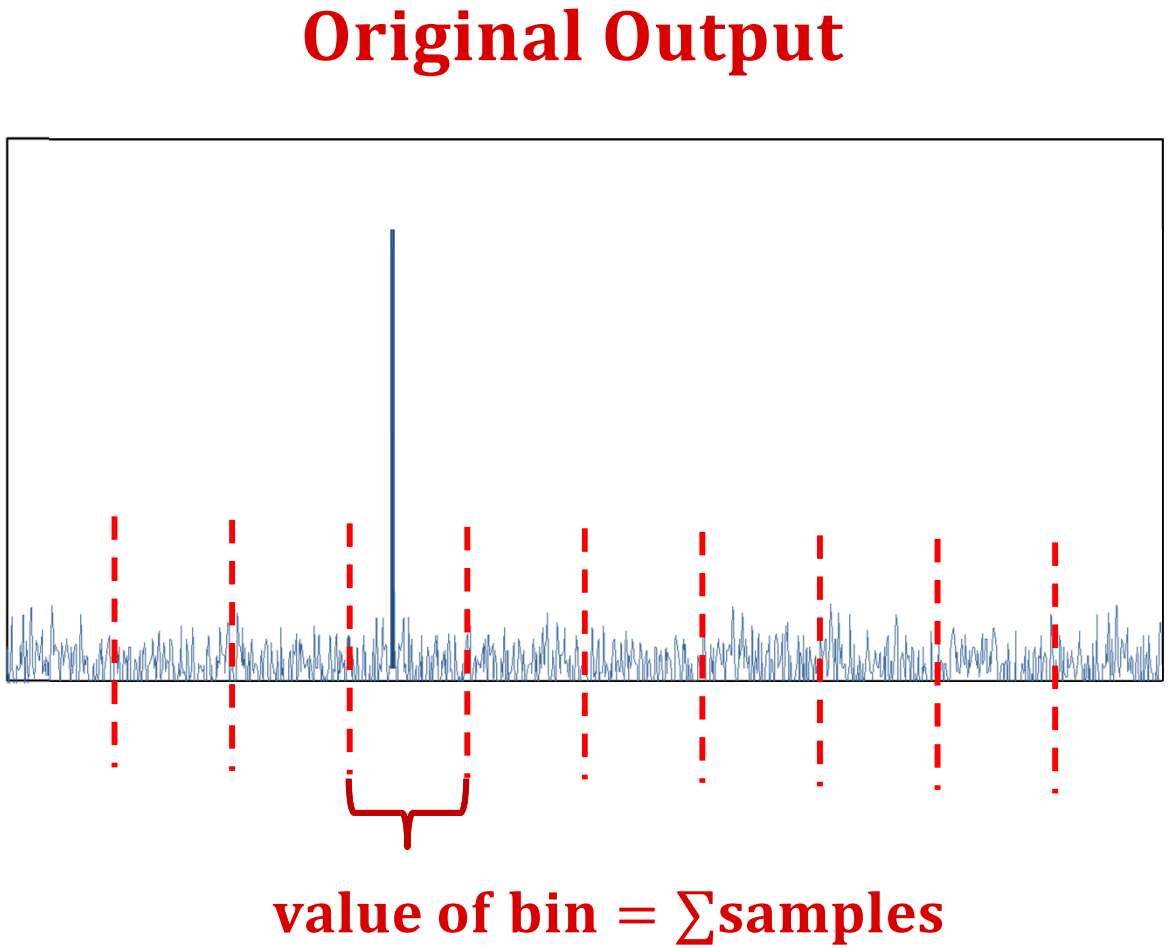
# Sparse IFFT for GPS

## 1- Binning

Divide output into a few bins

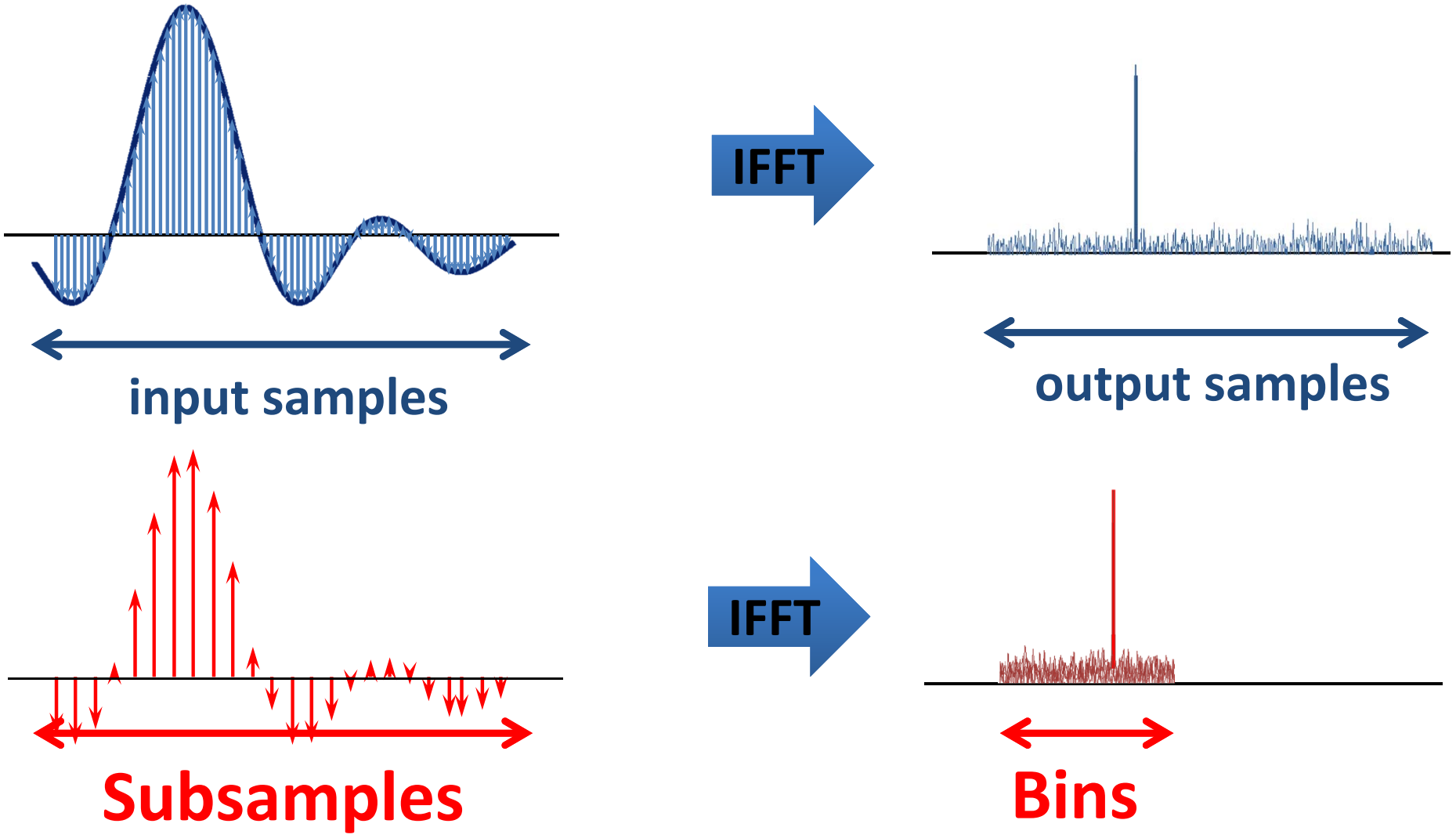
## 2- Estimation

Estimate the largest coefficient in the largest bin



# How to Do Binning Efficiently?

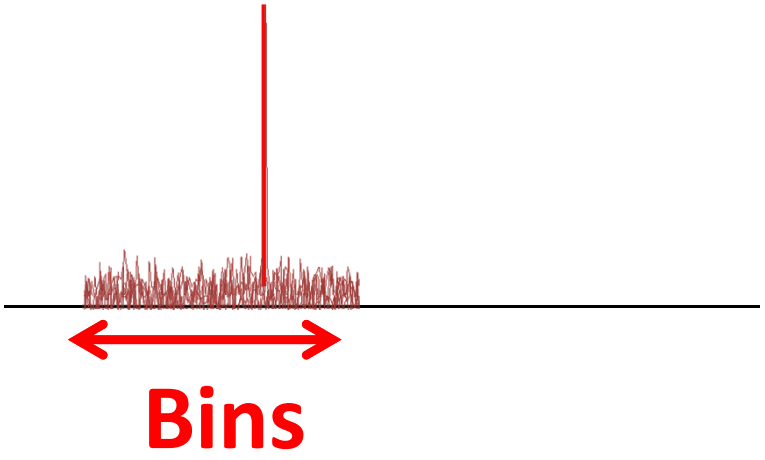
**Sub-sampling input➔ Aliasing of output**



# How to do Estimation Efficiently?

Out of the samples in the large bin, which one is the spike?

**The spike is the sample that has the maximum correlation**



# Sparse IFFT for GPS

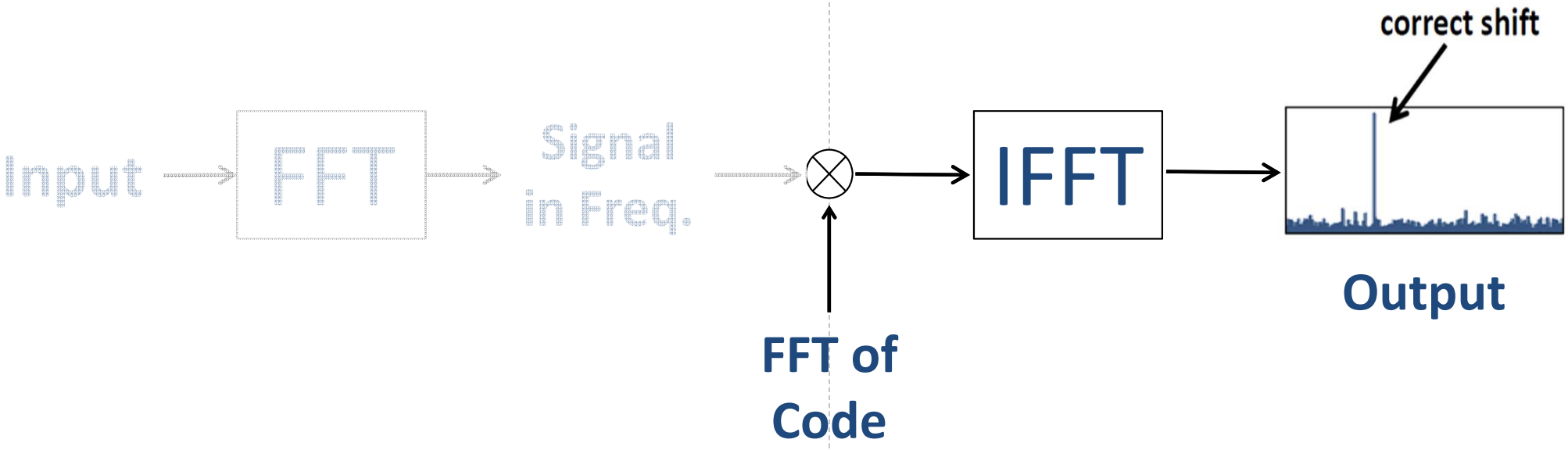
- $n$ is number of samples
- $l$ samples per bin → $n/l$ bins

**Binning:** $n/l \ \log(n/l)$

**Estimation:** $l \times n$

$$l = \sqrt{\log n} \quad \Rightarrow \quad O\left(n\sqrt{\log n}\right)$$

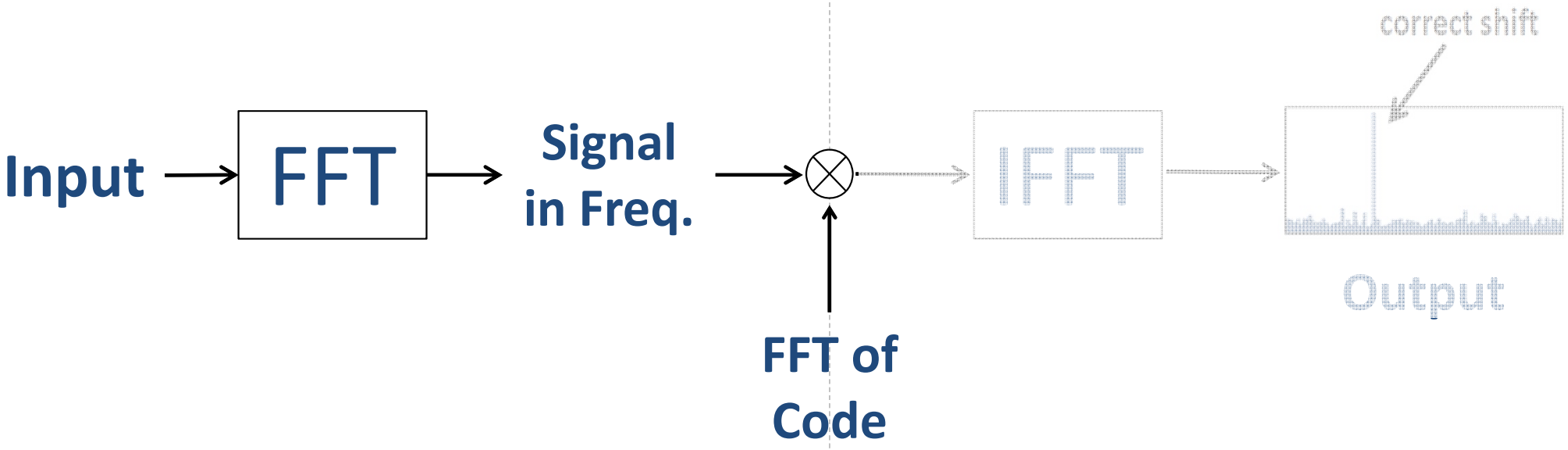
# GPS Using Sparse FFT




FFT Stage

**Sparse IFFT**

# GPS Using Sparse FFT



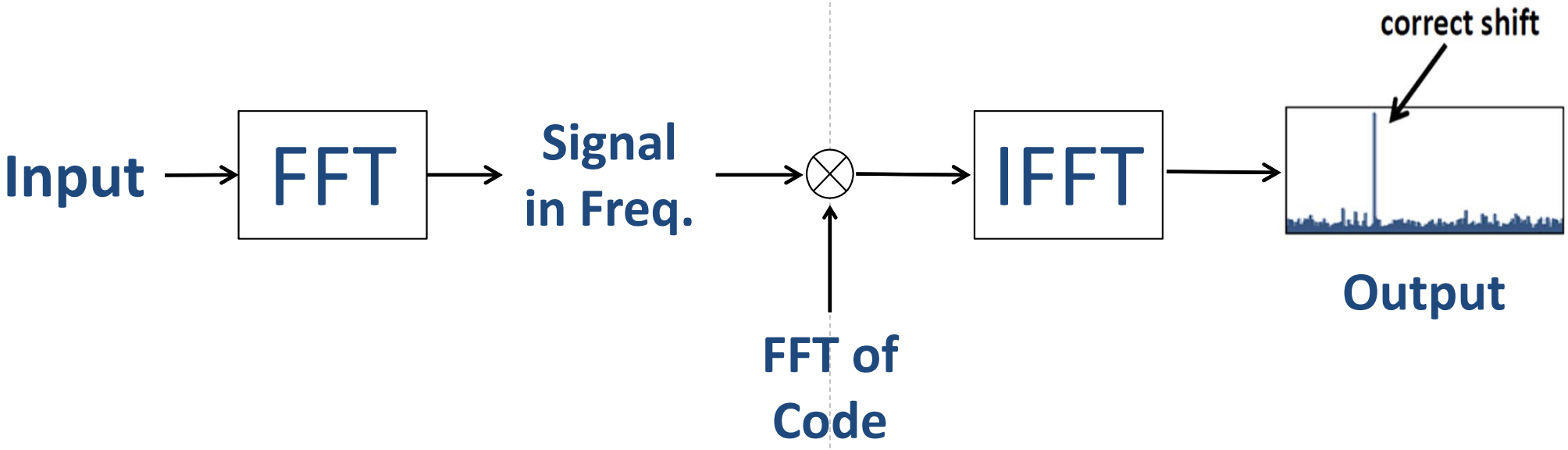

**FFT Stage**

Sparse IFFT

Output is not sparse

Cannot Use Sparse FFT

# GPS Using Sparse FFT

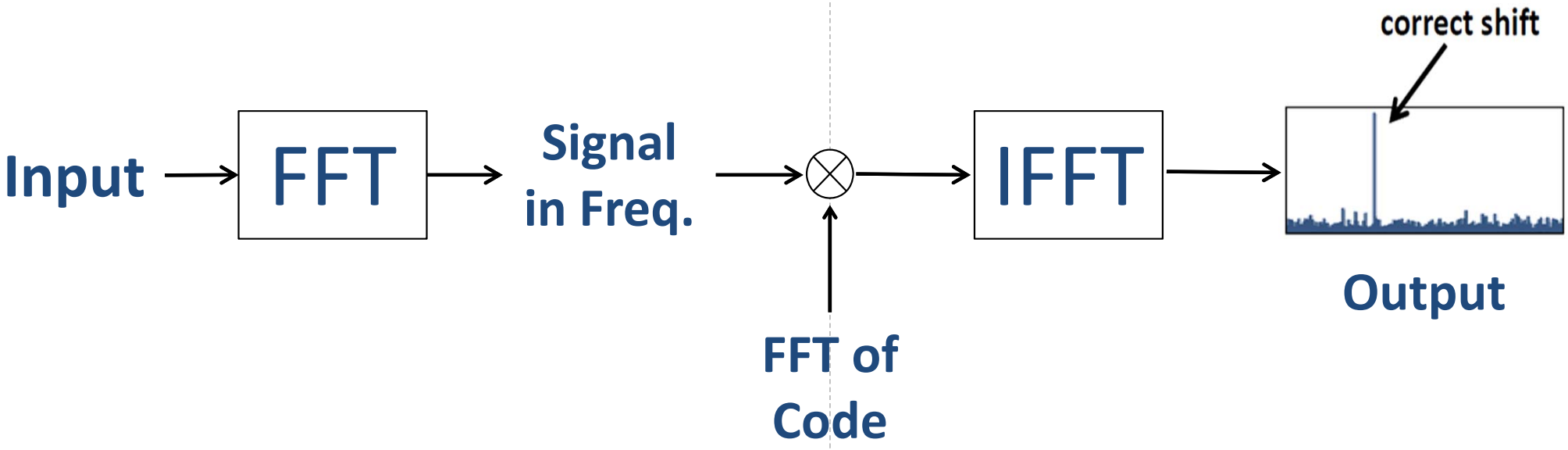



**FFT Stage**

**Sparse IFFT**

**Input to next stage**

# GPS Using Sparse FFT



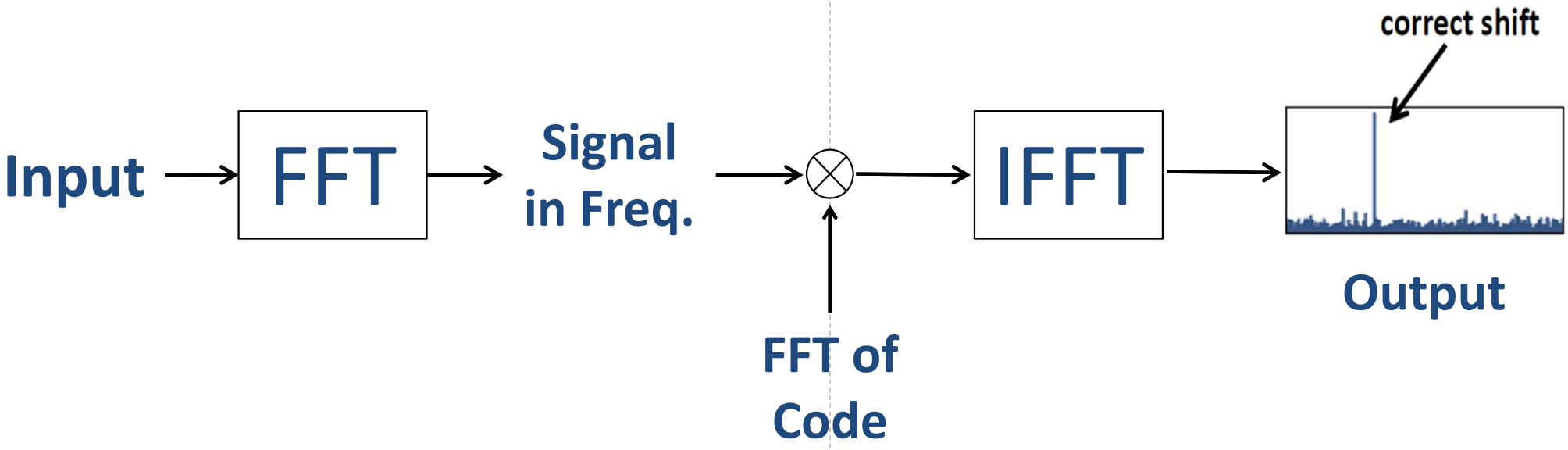

**Subsampled FFT**

**Need only few samples of FFT output**

**Sparse IFFT**

**IFFT samples its input**

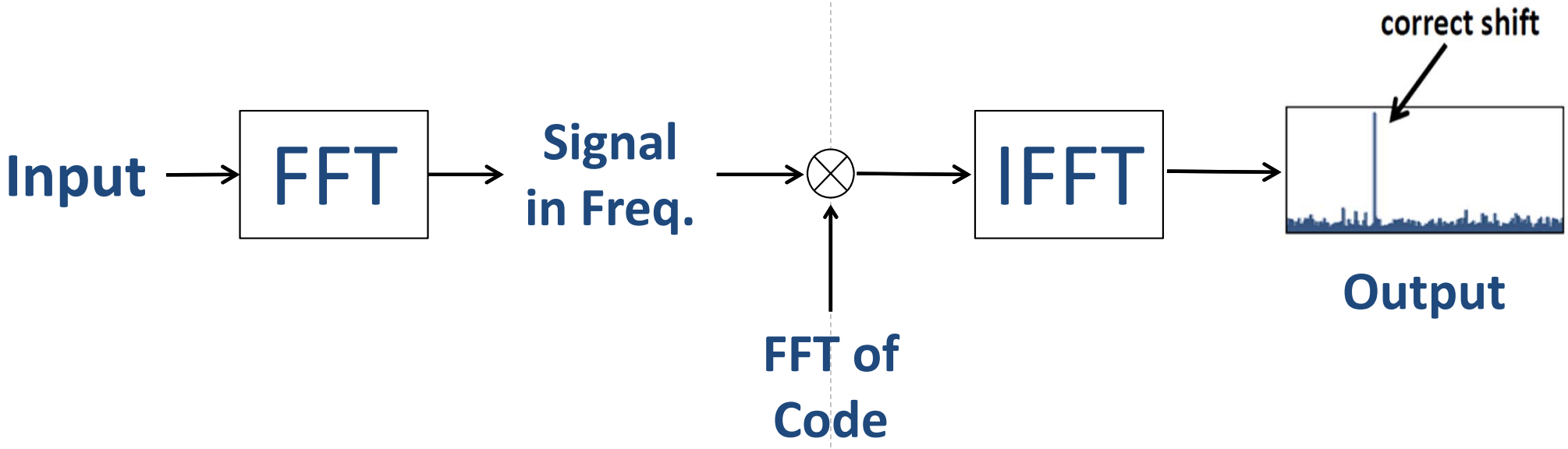
# GPS Using Sparse FFT




**Subsampled FFT** **Sparse IFFT**

FFT and IFFT are dual of each other

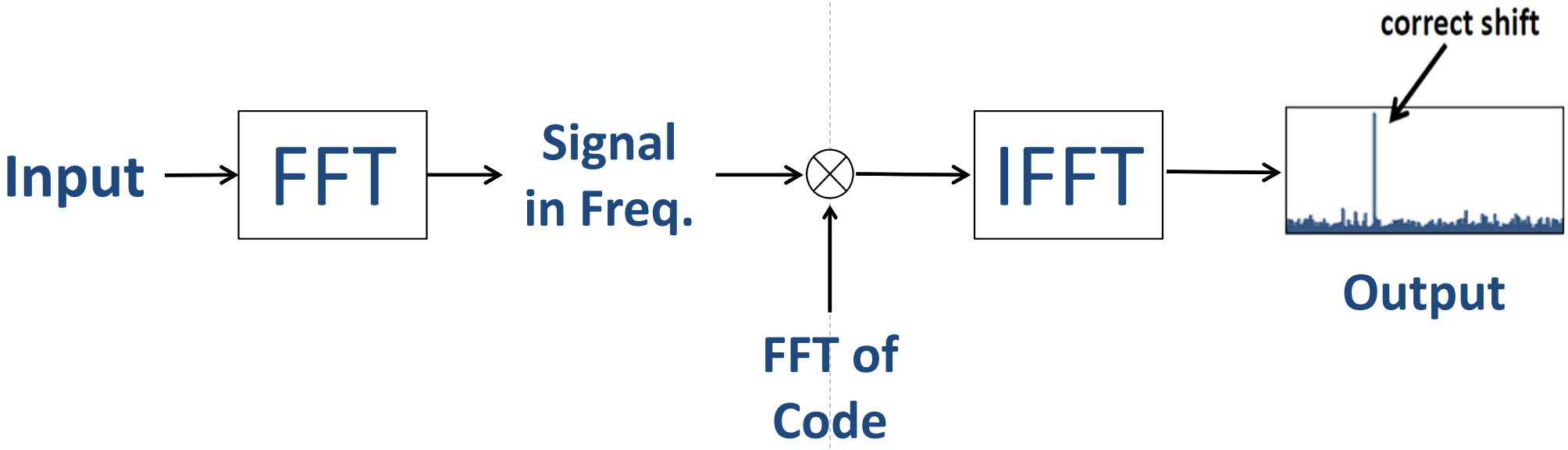
# GPS Using Sparse FFT




**Subsampled FFT** **Sparse IFFT**

**Aliasing input** → **FFT** → **Subsampling** → **IFFT** → **Aliasing output**

# GPS Using Sparse FFT



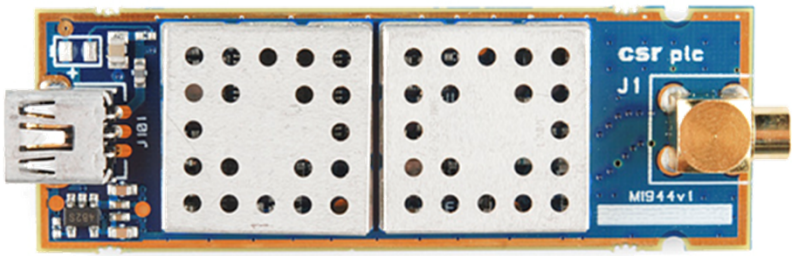

**Subsampled FFT**

$$O\left(n\sqrt{\log n}\right)$$

**Sparse IFFT**

$$O\left(n\sqrt{\log n}\right)$$

# Experiments



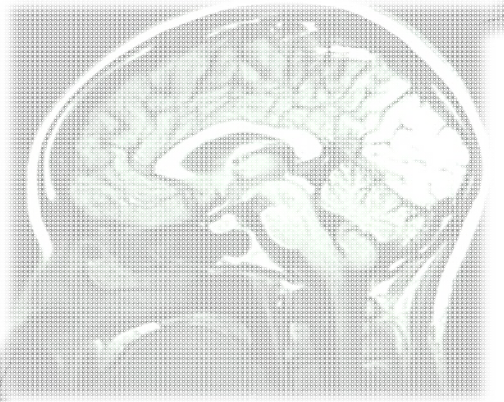


- Traces are collected both US and Europe
- $\text{Gain} = \dfrac{\text{Number Operations using FFT}}{\text{Number Operations using Sparse FFT}}$

**Sparse FFT provides a gain of 2.2x**

**Significant reduction in power**

# Applications of Sparse FFT



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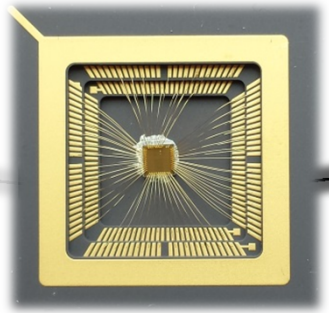
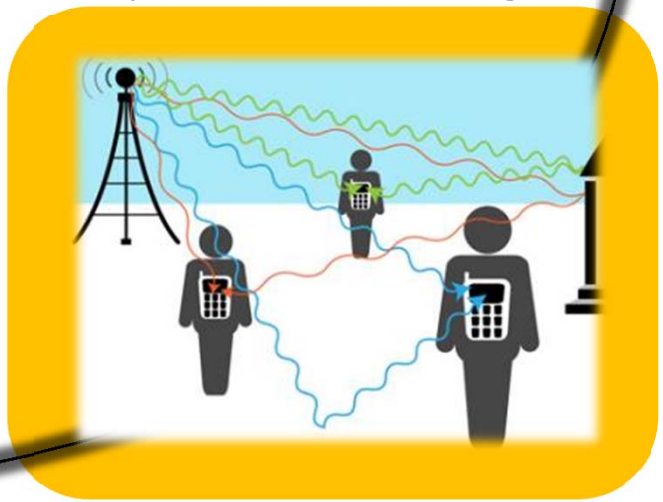
Computational Photography

Hardware



Spectrum Sensing



# Spectrum Shortage

- FCC : spectrum crunch started in 2013
- Increasing wireless demand

→But no more spectrum availability

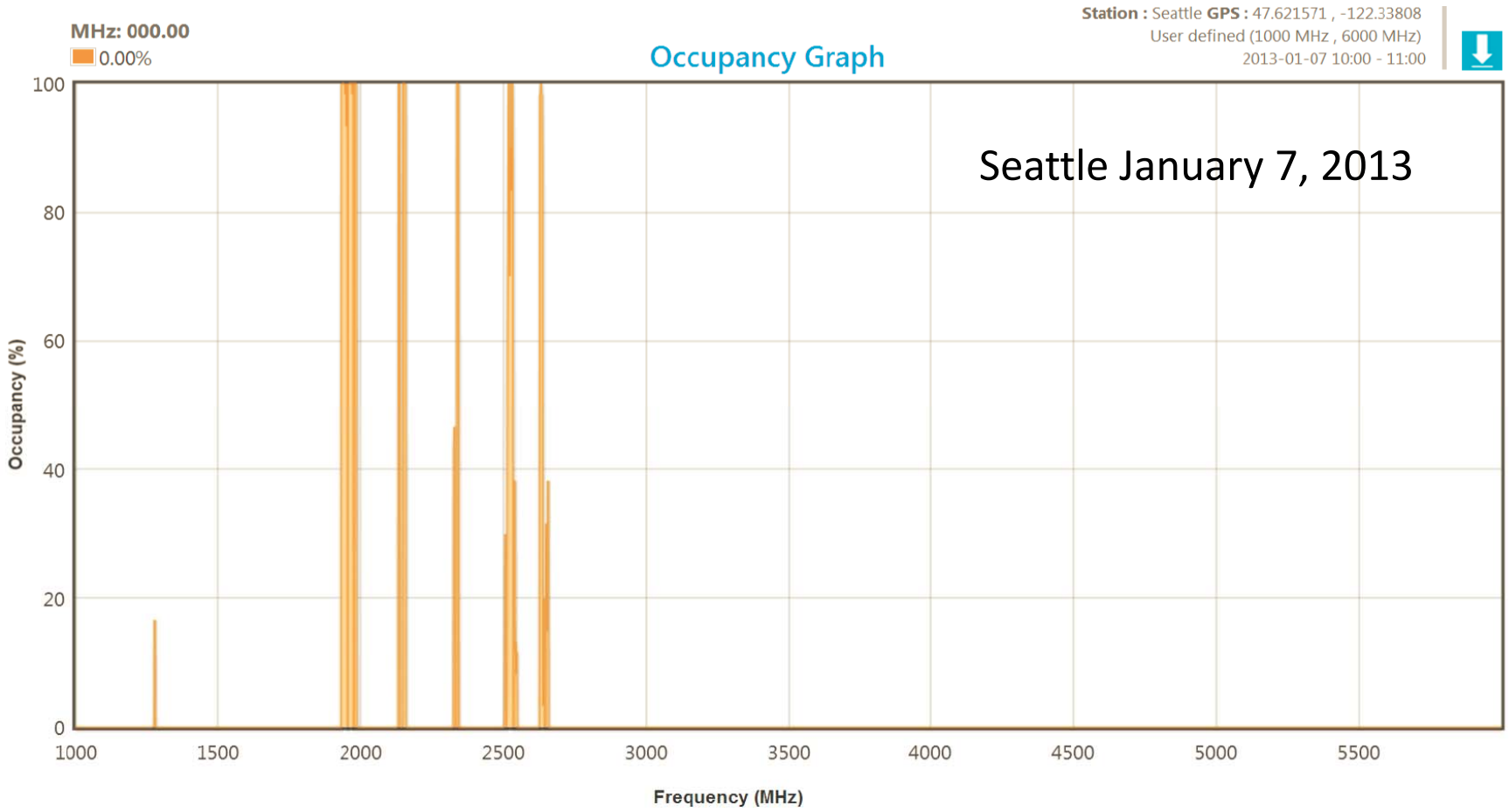
The iPhone 4 demo failed at Steve Jobs's keynote due to wireless congestion.

Jobs's reaction: *"If you want to see the demos, shut off your laptops, turn off all these MiFi base stations, and put them on the floor, please."*

H. Hassanieh, L. Shi, O. Abari, E. Hamed & D. Katabi, *GHz-Wide Sensing and Decoding Using the Sparse Fourier Transform,* **INFOCOM'12**

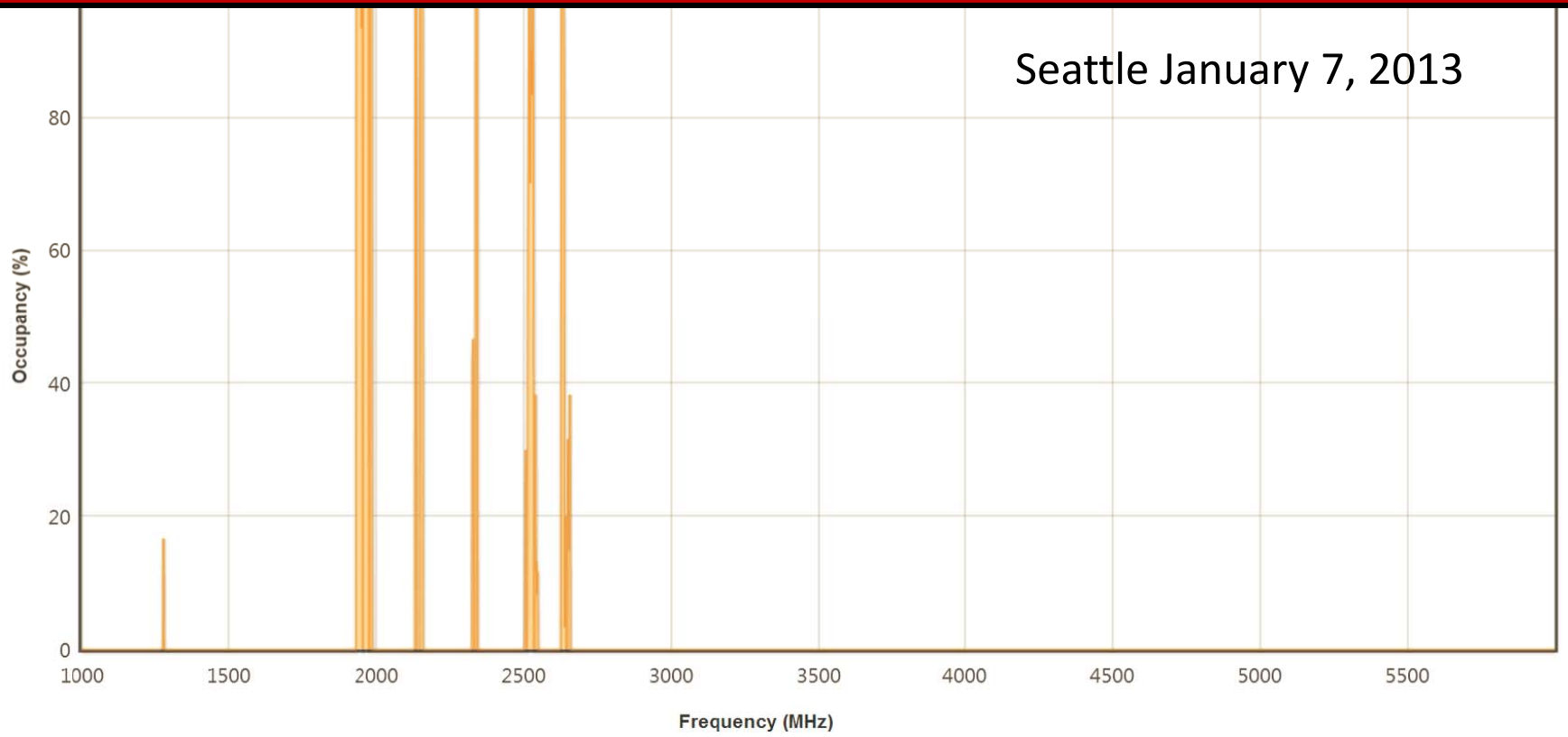
# Spectrum Shortage

But at any time, most of the spectrum is unused



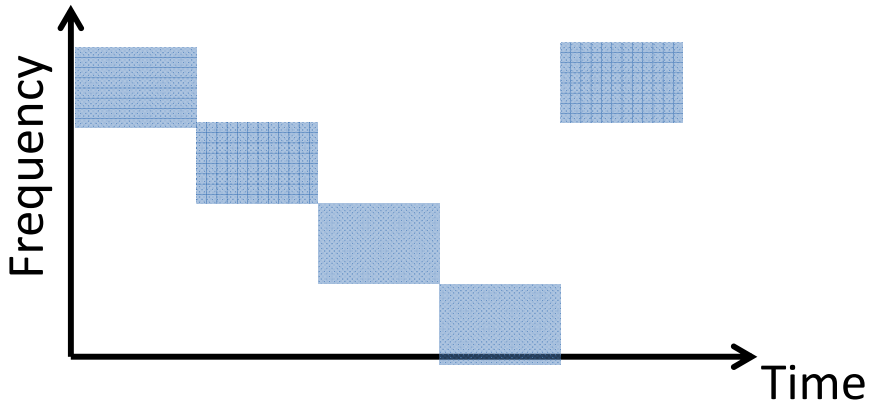
Solution: Sense to find unused bands → Use them!
Challenge: How to find unused bands with dynamic spectrum usage?

**GHz Spectrum Sensing in Real-time**



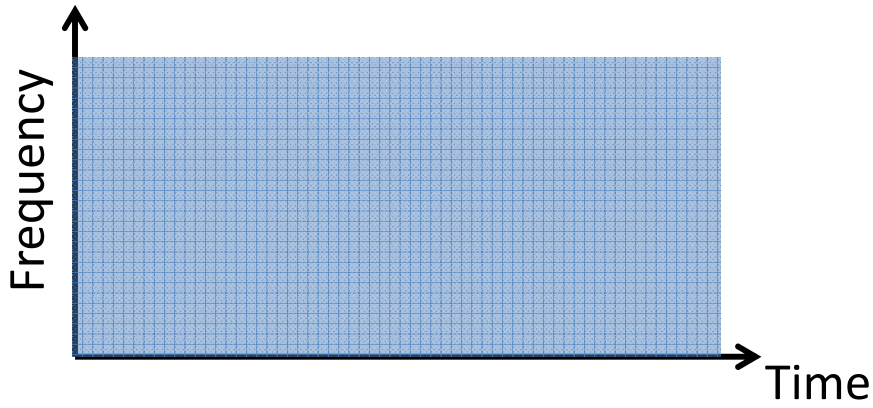
# Real-time GHz Spectrum Sensing is Difficult

**Today:**
**Sequential sensing with MHz Sampling**




- Cheap, low power
- Miss important signals

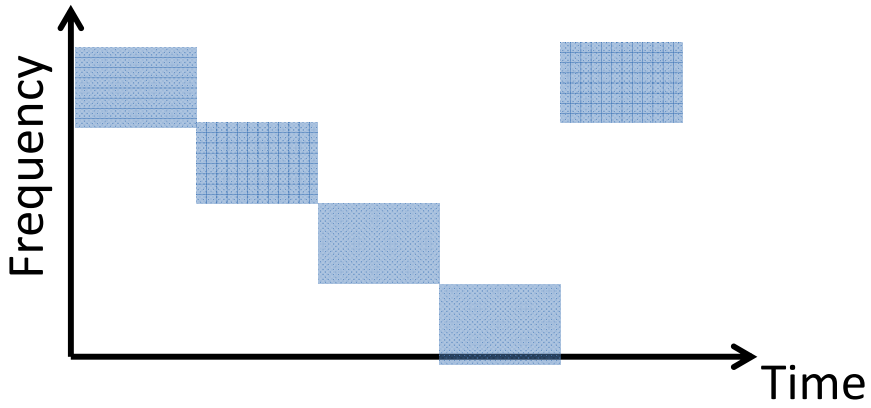
**Ideally:**
**Real-time sensing with GHz Sampling**



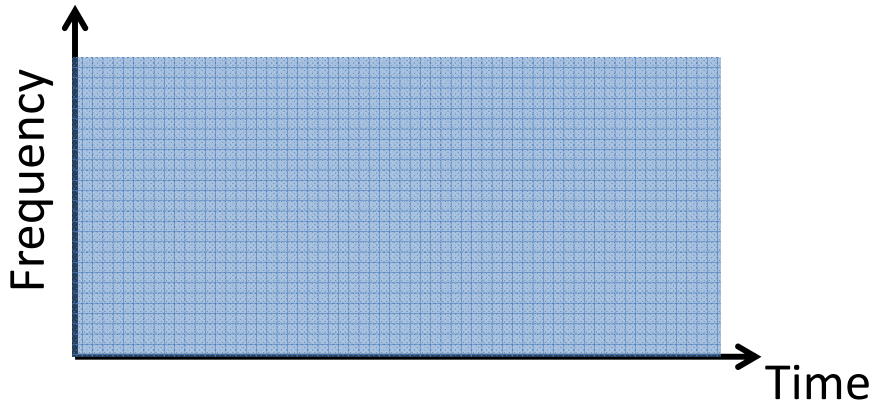

- Capture all signals
- Costly, high power, and effectively impractical

# Real-time GHz Spectrum Sensing is Difficult

**Today:**
**Sequential sensing with MHz Sampling**

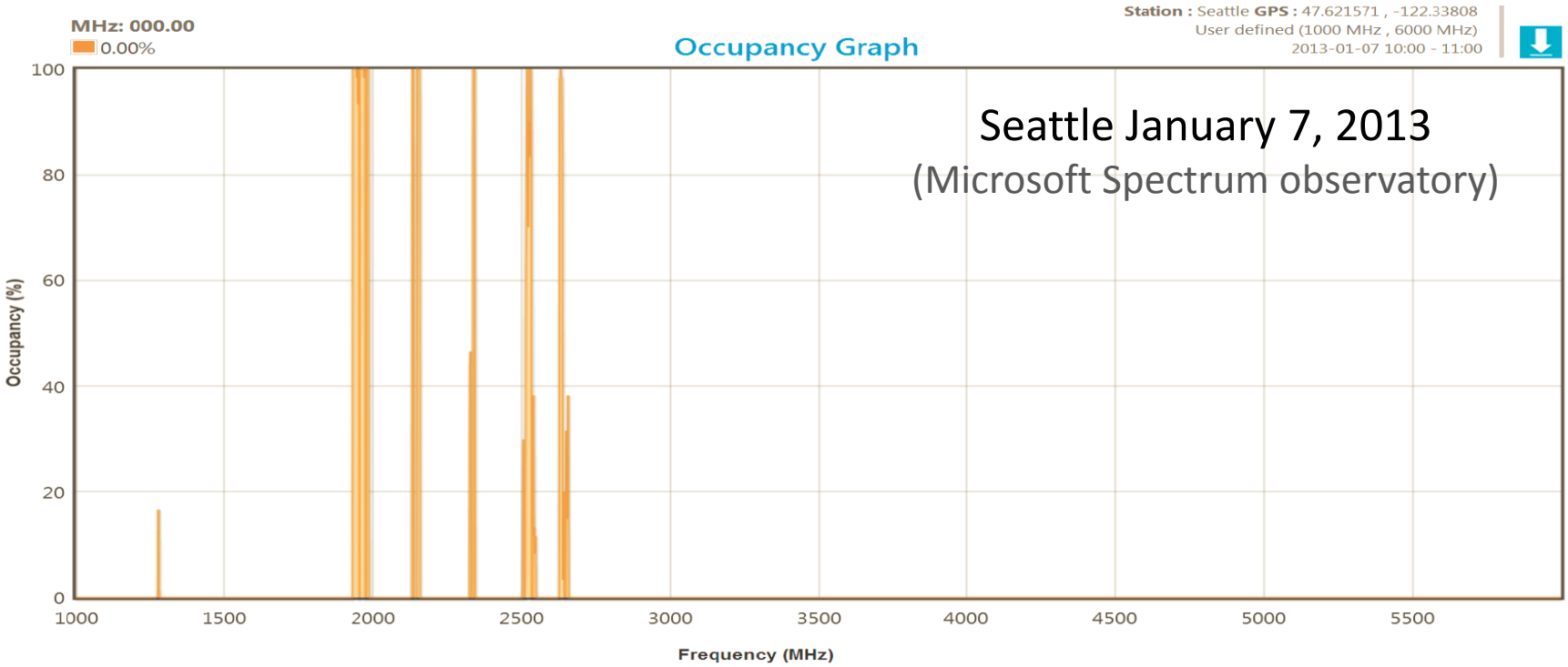



**Ideally:**
**Real-time sensing with GHz Sampling**




Goal:
Real-time GHz sensing with MHz sampling

# Leverage the Spectrum Sparsity




**Use Sparse FFT**

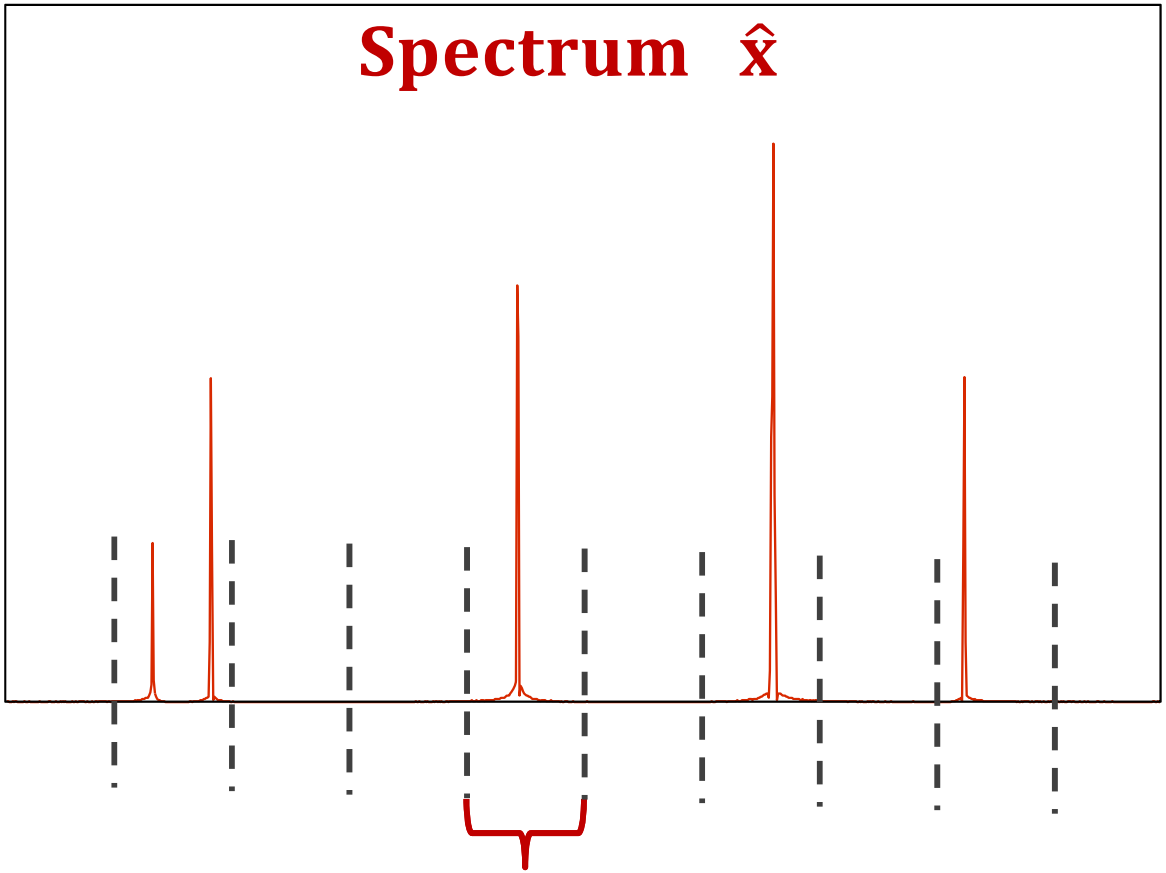
# Sparse FFT for Spectrum Sensing

## 1- Binning

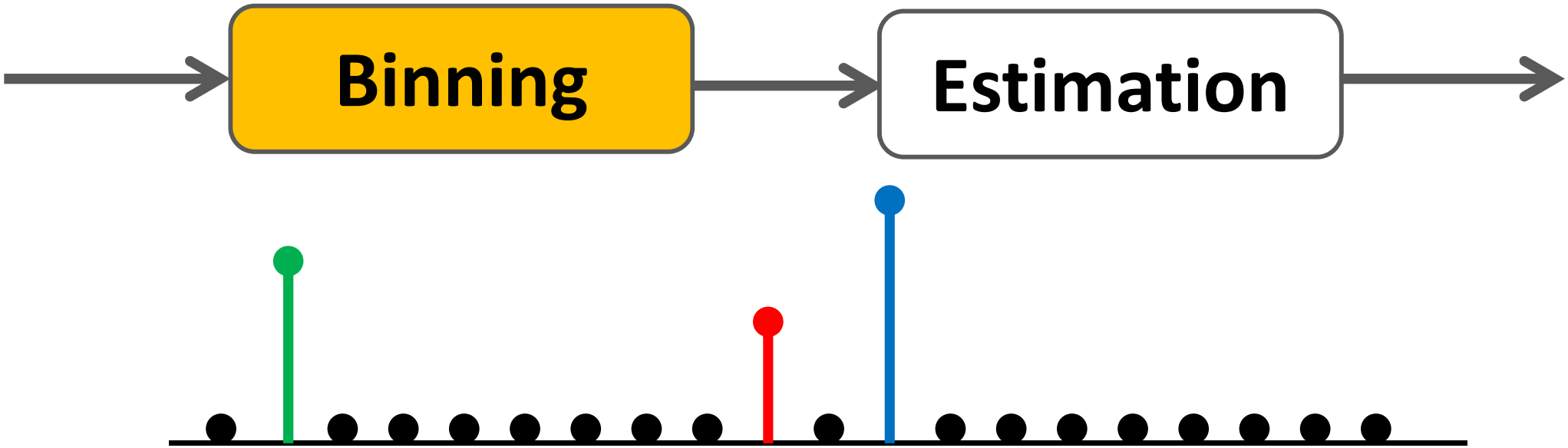
Divide spectrum into a few bins

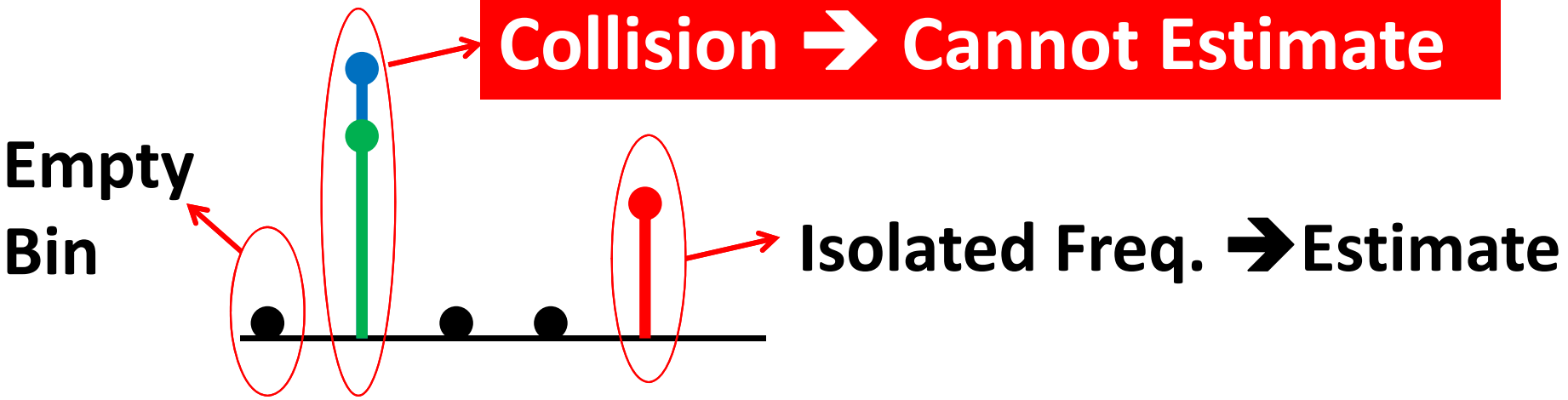
## 2- Estimation

Estimate the large coefficients in each non-empty bin

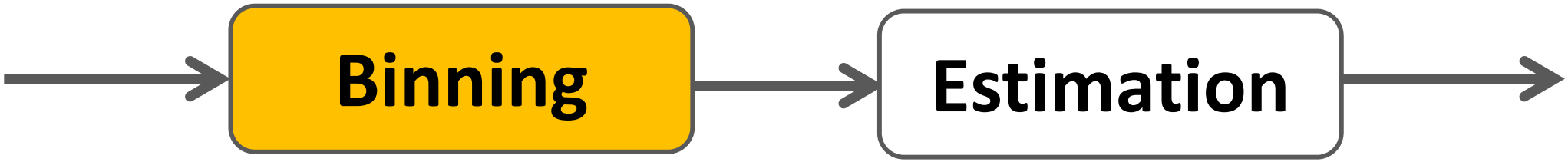


# Sparse FFT for Spectrum Sensing



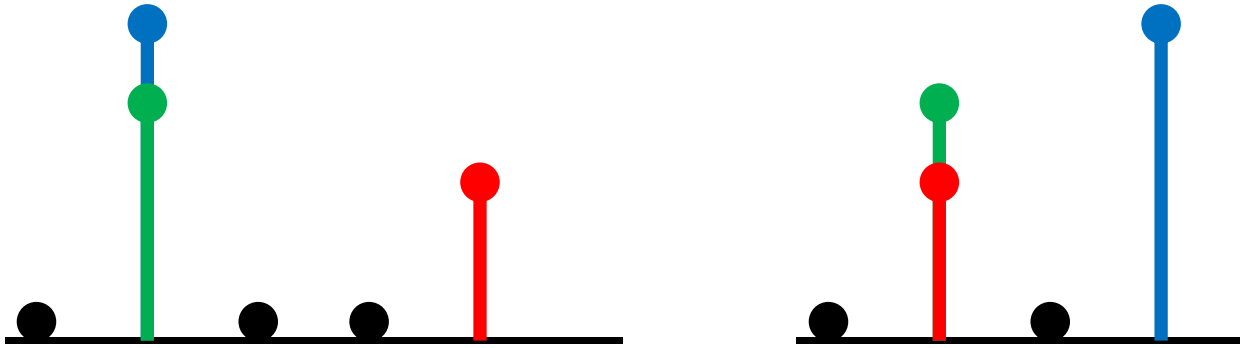



# Sparse FFT for Spectrum Sensing

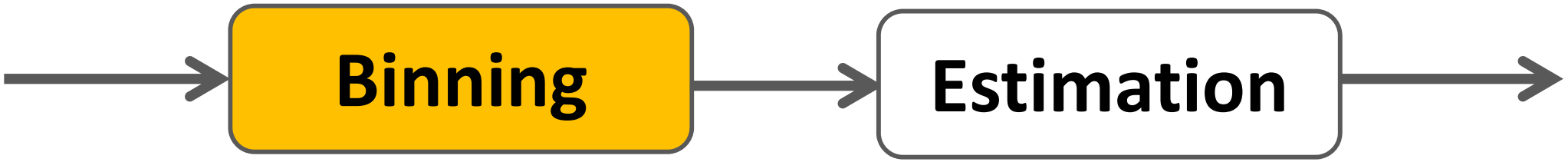



Bin multiple time using **co-prime sub-sampling**

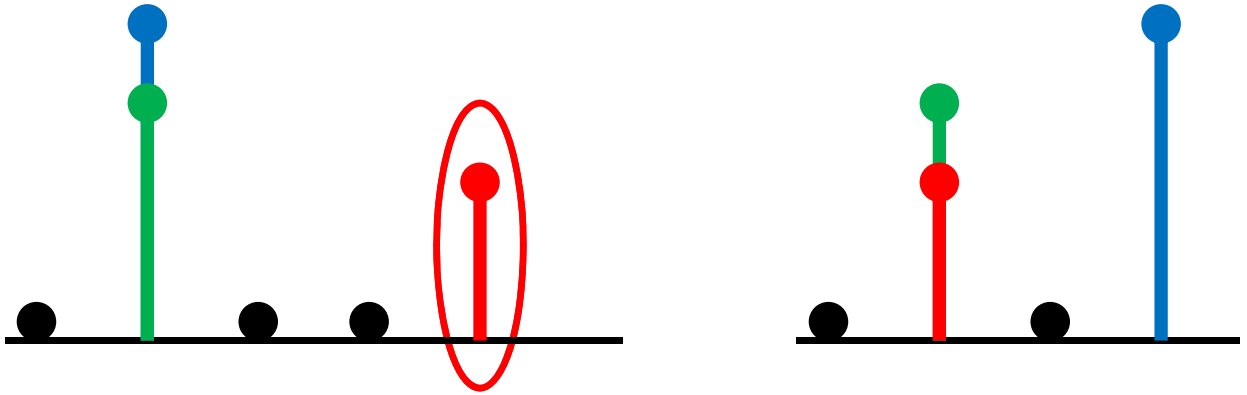
Same frequencies don’t collide in both



# Sparse FFT for Spectrum Sensing

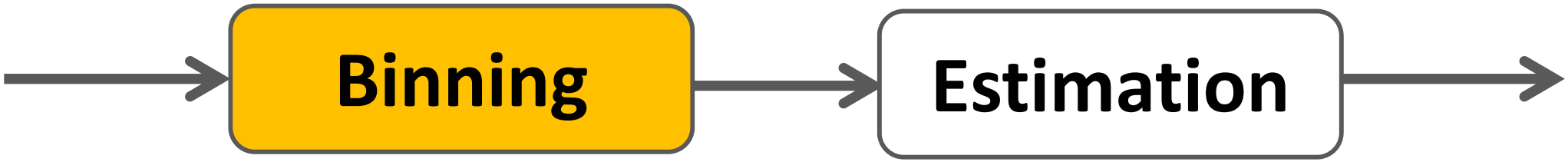



Identify isolated freq. in one binning and subtract them from the other; and iterate …



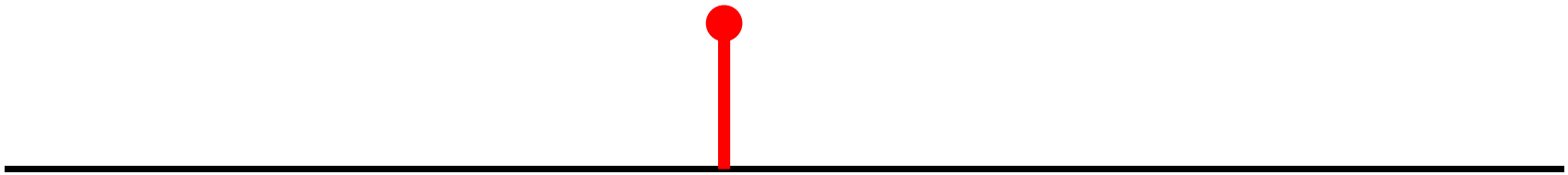
**Output Result:**

# Sparse FFT for Spectrum Sensing

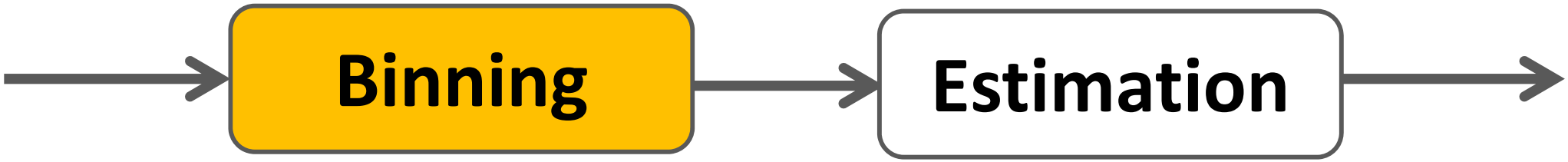



Identify isolated freq. in one binning and subtract them from the other; and iterate …

**Output Result:**



# Sparse FFT for Spectrum Sensing

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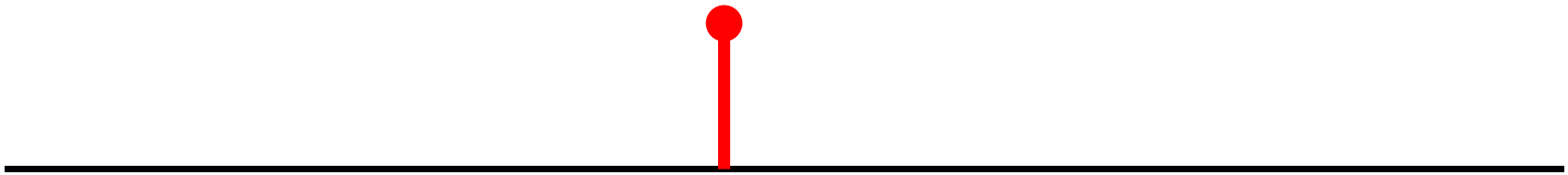
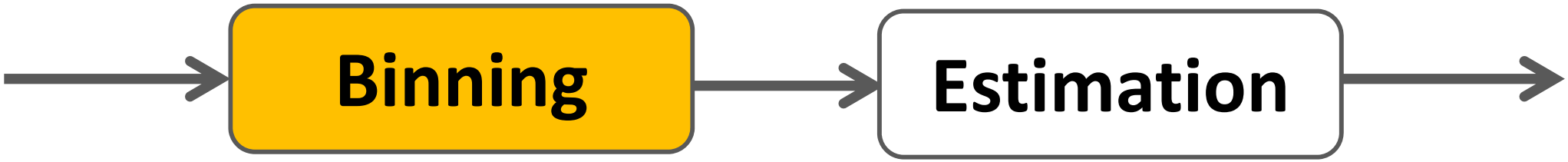


# Sparse FFT for Spectrum Sensing

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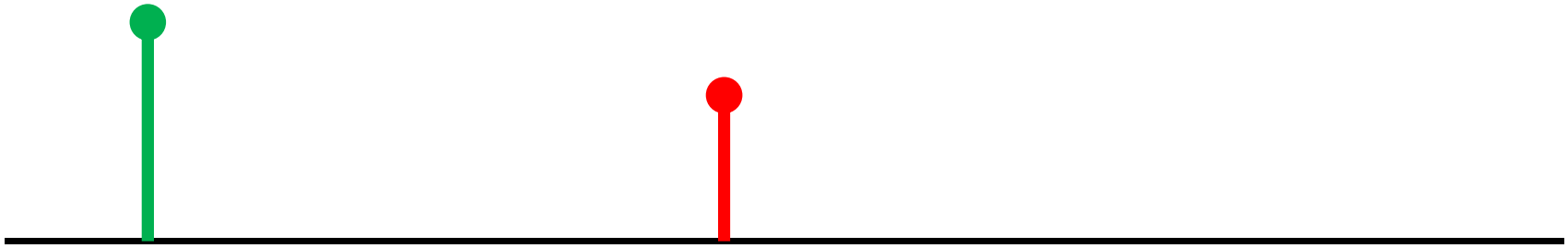
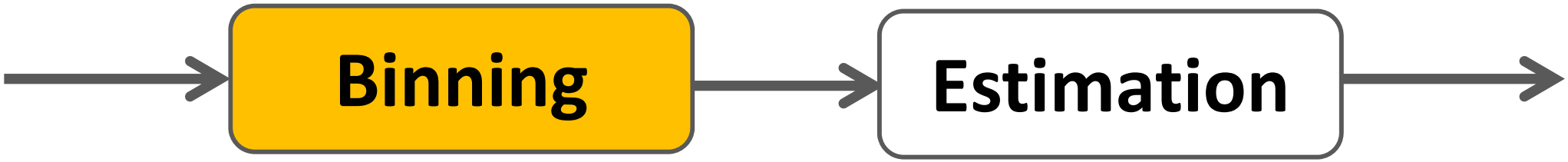
**Output Result:**



# Sparse FFT for Spectrum Sensing

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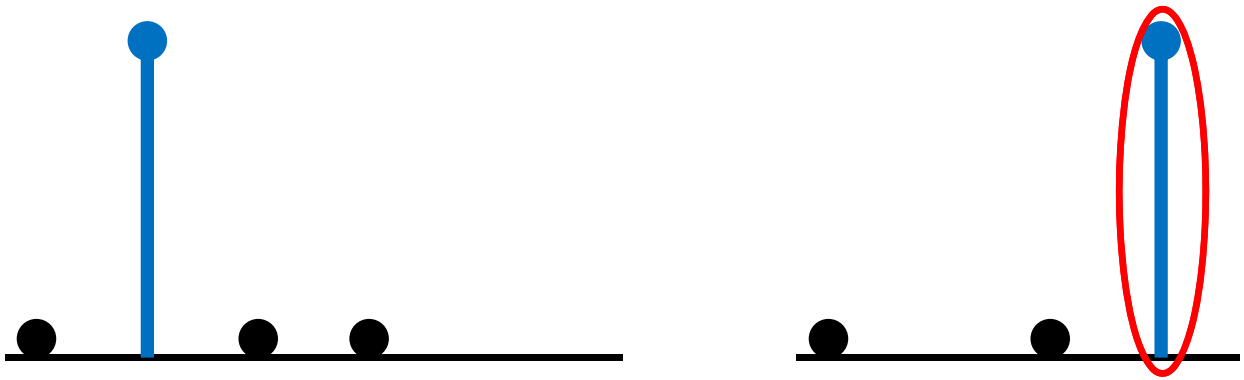
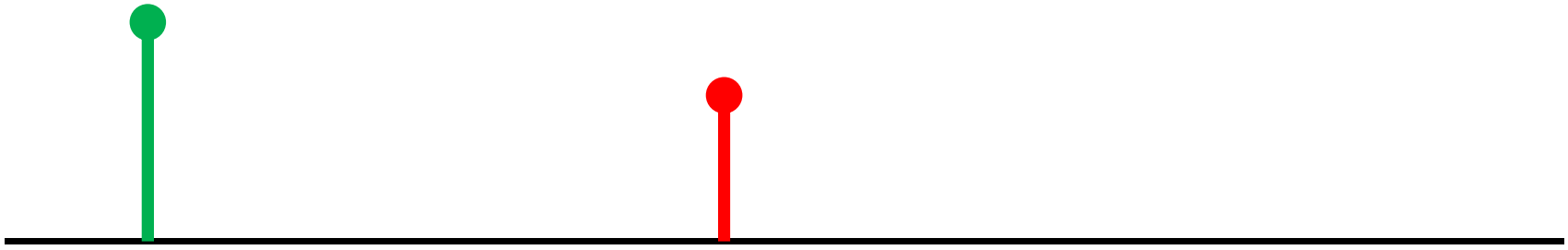
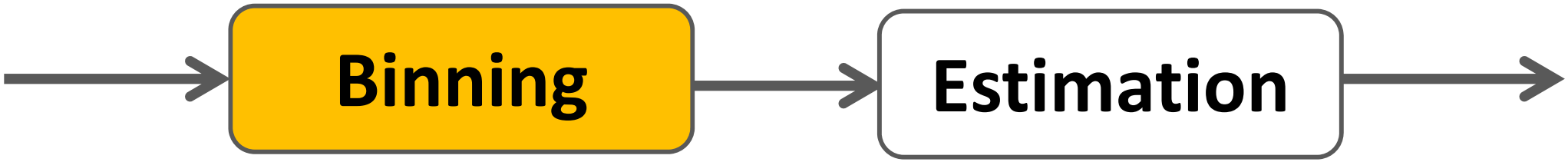


**Output Result:**



# Sparse FFT for Spectrum Sensing

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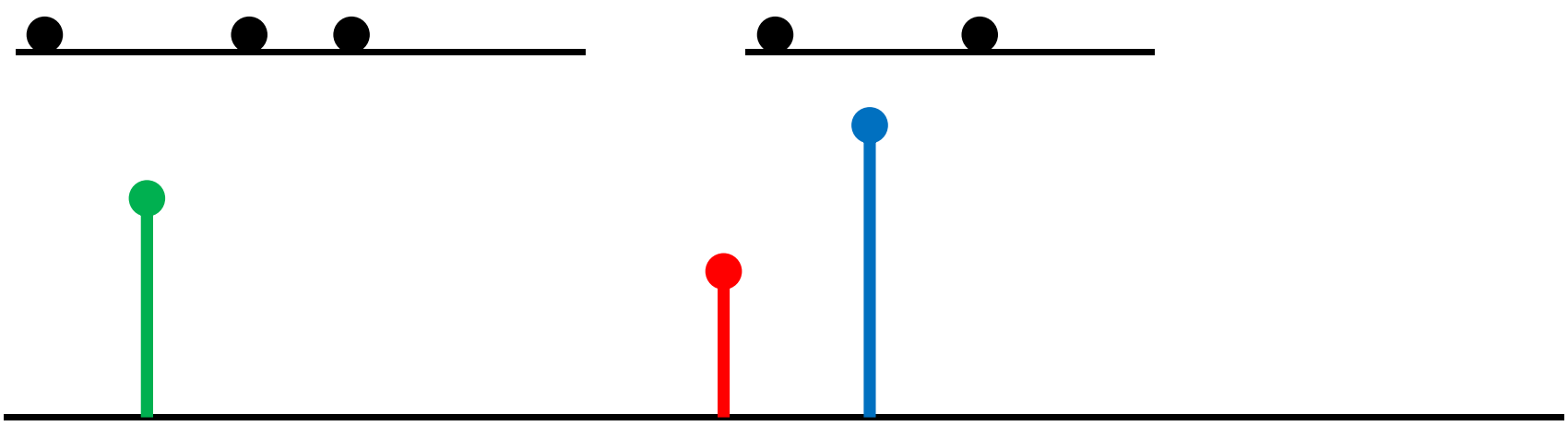
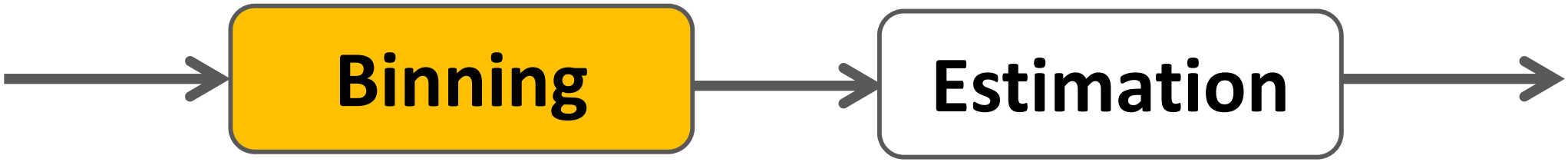


# Sparse FFT for Spectrum Sensing

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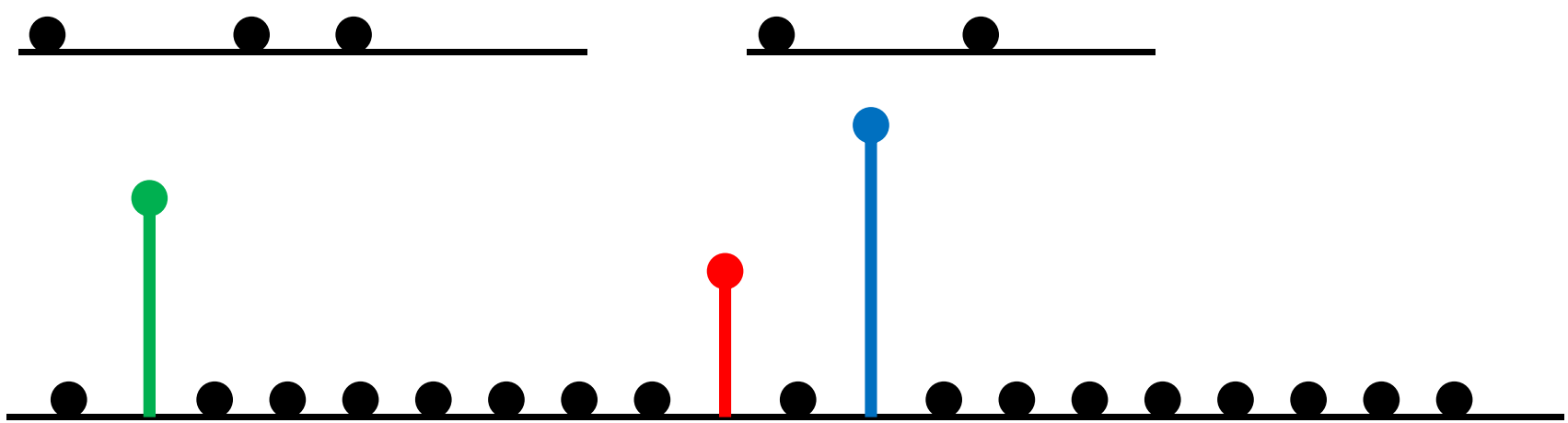
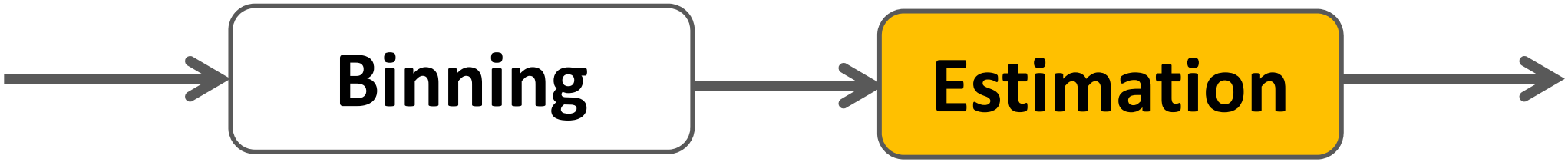


# Sparse FFT for Spectrum Sensing




Repeat binning after shifting the signal in time by a **time shift $\tau$**

| **Time-Domain** | **Freq-Domain** |
|---|---|
| $x(t)$ | $X(f)$ |
| $x(t-\tau)$ | $X(f)e^{-j\theta}$ |

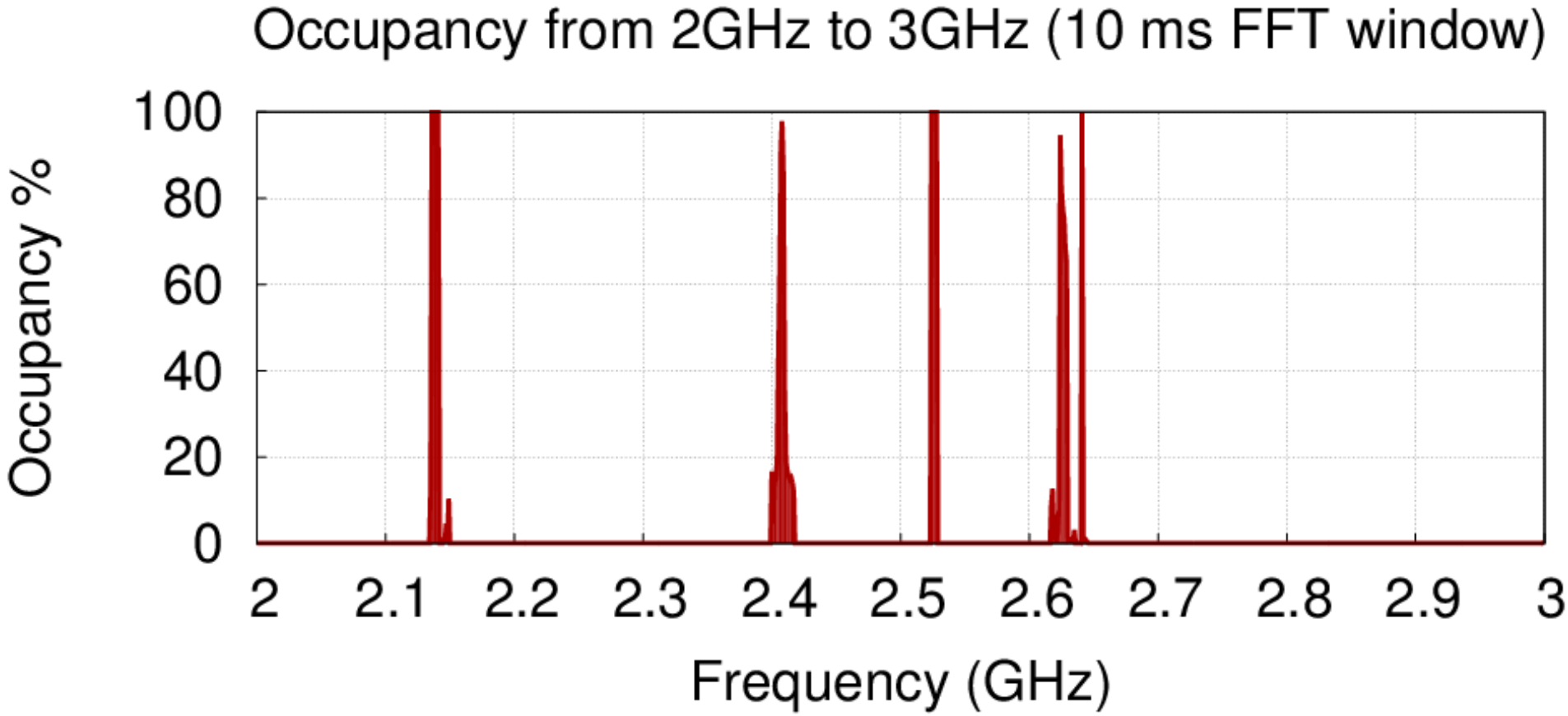
Phase Rotation : $\theta = \frac{2\pi f \tau}{N} \rightarrow f = \frac{N\theta}{2\pi\tau}$

# GHz Receiver Using the Sparse FFT

**Built a receiver that can capture GHz of spectrum using a total of 150 MHz sampling rate**

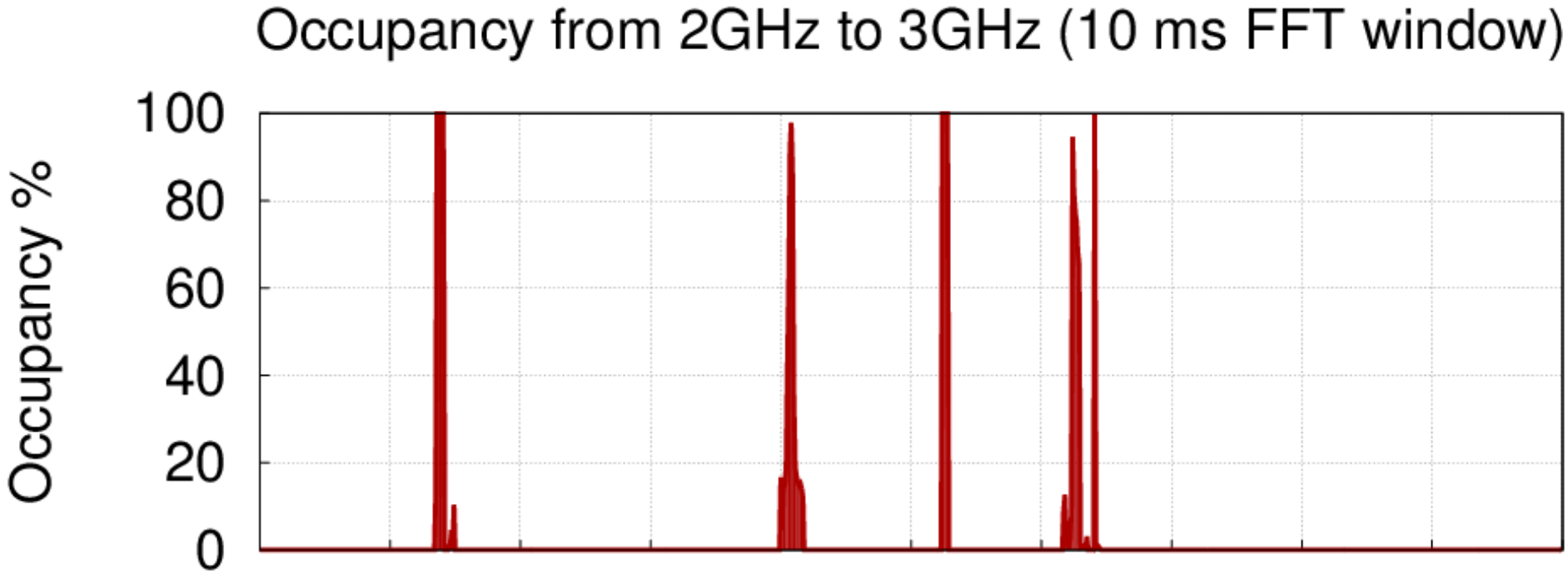
# GHz Receiver Using the Sparse FFT

Cambridge, MA January 15 2013



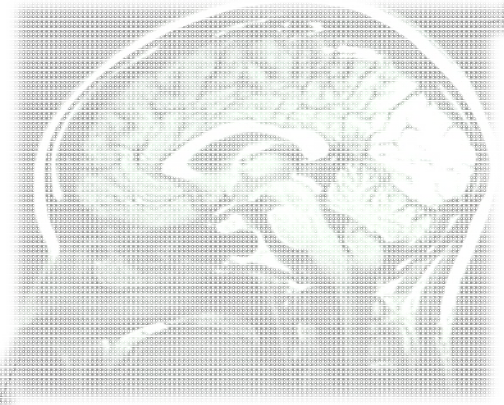
# GHz Receiver Using the Sparse FFT

Cambridge, MA January 15 2013




SFFT enables realtime GHz sensing for low-power portable devices

# Applications of Sparse FFT

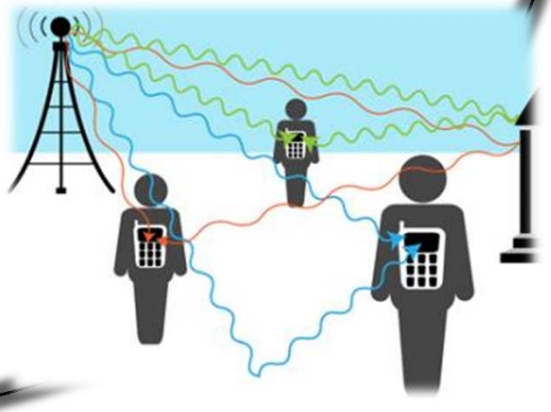


Medical Imaging

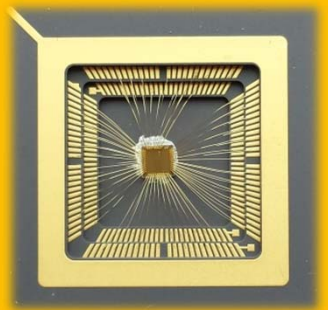
GPS

Computational Photography

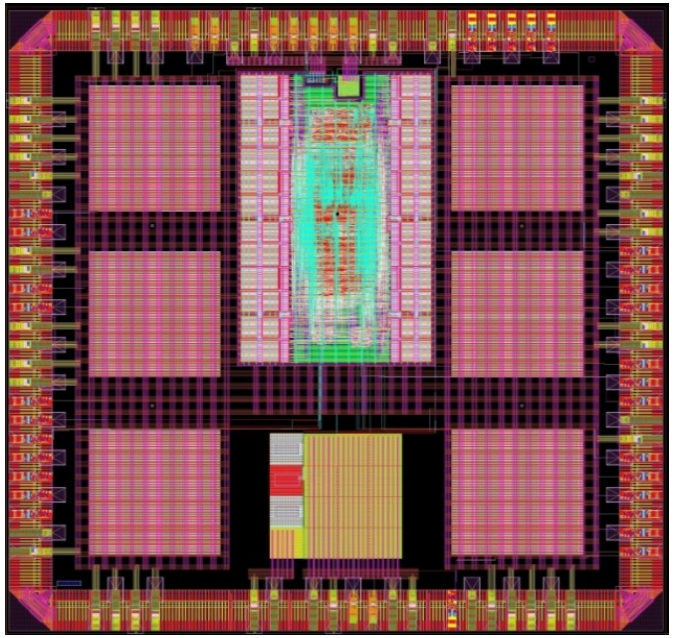
Spectrum Sensing

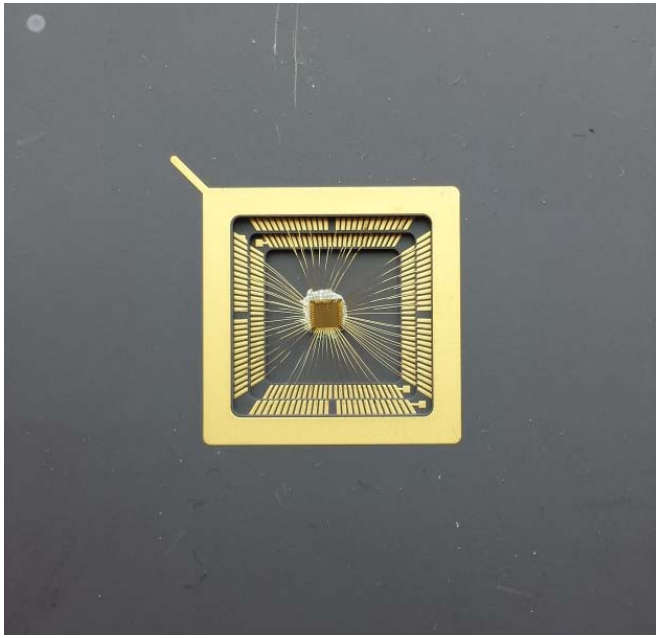


Hardware



# 0.75-Million Point Sparse FFT Chip

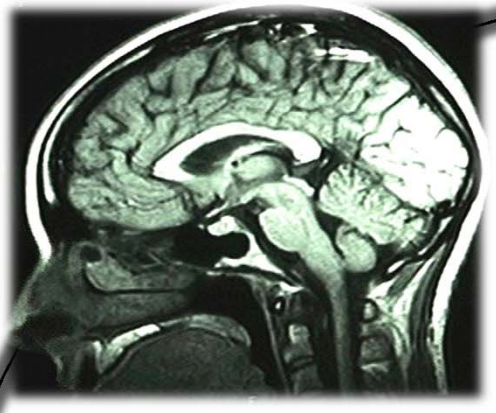




| | **Traditional FFTs** [ISSCC'11, JSSC'08, JSSC'12] | **Sparse FFT** [ISSCC'14] |
|---|---|---|
| **Size** | ≈ Thousand Point | **≈ Million Point** |
| **Energy/Sample** | 17.2pJ | **0.4pJ (40x lower power)** |

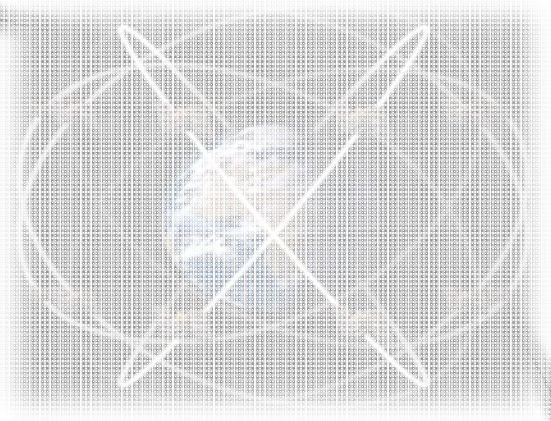
O. Abari, E. Hamed, H. Hassnanieh, A. Agarwal, D. Katabi, A. Chandrakasan, V. Stojanovic, *A 0.75 Million-Point Fourier Transform Chip for Frequency Sparse Signals* **ISSCC'14**

# Applications of Sparse FFT



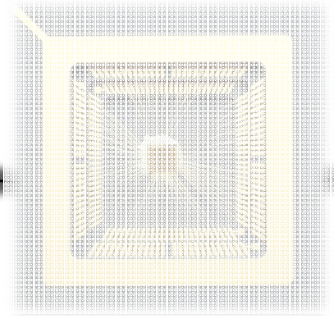
Medical Imaging

GPS



Computational Photography

Hardware



Spectrum Sensing